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Partial Fractions

Recognition Training · Set III

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 5 — Calculus
Level	Medium → Very Hard
Questions	6
Total marks	32
Instructions	Show all working. **M1** = method mark (correct process). **A1** = accuracy mark (correct result). **R1** = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

Partial fractions is a *two-stage* technique. **Stage 1:** Decompose the rational expression into its partial fraction form. **Stage 2:** Integrate each fraction separately. These are two distinct cognitive tasks and must be treated as such. Students who rush through Stage 1 to reach the integration carry errors in their constants (A, B, C) through every subsequent line. Complete the decomposition fully and verify it by recombining before beginning Stage 2. The most common error in IB AA HL partial fractions is not the integration — it is an incorrect constant in the decomposition.

Question 1 — Medium — [5 marks]

Find the following integral.

$$\int \frac{1}{(x+1)(x+3)}\,dx$$

MISTAKE ANALYSIS

Distinct linear factors. Write $\dfrac{1}{(x+1)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+3}$*. Students rush to set* $x=-1$ *and* $x=-3$ *but make arithmetic errors substituting negative values. After finding* A *and* B*, always verify: recombine the partial fractions and confirm they equal the original expression before integrating.*

Question 2 Medium [5 marks]

Find the following integral.

$$\int \frac{3x+1}{(x-1)(x+2)}\,dx$$

MISTAKE ANALYSIS

Distinct linear factors with a non-trivial numerator. Write $\frac{3x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$*. Students frequently confuse the sign of* B *when substituting* $x=-2$*: note that* $3(-2)+1=-5$ *and* $(x-1)\big|_{x=-2}=-3$*, giving* $B = \frac{-5}{-3} = \frac{5}{3}$*. Negative divided by negative is positive — a common sign error.*

Question 3 Medium–Hard [6 marks]

Find the following integral.

$$\int \frac{5x-2}{(x+1)(x-2)(x+3)}\,dx$$

MISTAKE ANALYSIS

Three distinct linear factors. Write $\frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x+3}$*. This requires three separate substitutions:* $x=-1$*,* $x=2$*,* $x=-3$*. Students find* A *and* B *correctly then make an arithmetic error on* C *due to fatigue on the third substitution. Work methodically: one substitution at a time, show all arithmetic, do not skip steps when the numbers are negative.*

Question 4 Hard [6 marks]

Find the following integral.

$$\int \frac{4x}{(x+1)^2(x-1)}\,dx$$

MISTAKE ANALYSIS

Repeated linear factor. The correct decomposition is $\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$*. Students incorrectly write*

$\dfrac{A}{x+1}+\dfrac{B}{x+1}+\dfrac{C}{x-1}$, combining the two terms for the repeated factor into one. A repeated factor $(x+a)^2$ requires both $\dfrac{A}{x+a}$ and $\dfrac{B}{(x+a)^2}$ in the decomposition. This is non-negotiable.

Question 5 Hard [5 marks]

Evaluate the following definite integral.

$$\int_0^2 \frac{1}{(x+1)(x+3)}\,dx$$

MISTAKE ANALYSIS

This is Q1 converted to a definite integral. The partial fraction decomposition is identical. The error occurs at the evaluation stage: students apply $\ln|x+1|$ and $\ln|x+3|$ correctly but substitute the limits into each logarithm separately before combining, introducing arithmetic errors. Write the full expression $\left[\frac{1}{2}\ln|x+1|-\frac{1}{2}\ln|x+3|\right]_0^2$ and use logarithm laws to simplify before substituting.

Question 6 Very Hard [5 marks]

Find the following integral.

$$\int \frac{x^2+2}{(x^2+1)(x+1)}\,dx$$

MISTAKE ANALYSIS

Irreducible quadratic factor. The correct decomposition is $\dfrac{Ax+B}{x^2+1}+\dfrac{C}{x+1}$. Students who write $\dfrac{A}{x^2+1}+\dfrac{B}{x+1}$ lose the Ax term in the numerator — a fundamental error that produces an incorrect system of equations. An irreducible quadratic factor x^2+1 always requires a linear numerator $Ax+B$, not a constant. After decomposition, $\int \frac{Ax+B}{x^2+1}dx$ splits into $\int \frac{Ax}{x^2+1}dx+\int \frac{B}{x^2+1}dx$, giving $\frac{A}{2}\ln(x^2+1)+B\arctan(x)$.

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution — Question 1

Decompose	$\dfrac{1}{(x+1)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x+3}$	M1
Find A: $x=-1$	$1 = A(2) \;\Rightarrow\; A = \frac{1}{2}$	A1
Find B: $x=-3$	$1 = B(-2) \;\Rightarrow\; B = -\frac{1}{2}$	A1
Integrate	$\frac{1}{2}\ln\lvert x+1\rvert - \frac{1}{2}\ln\lvert x+3\rvert + C$	M1
Simplify	$\frac{1}{2}\ln\left\lvert\dfrac{x+1}{x+3}\right\rvert + C$	A1

Final answer: $\dfrac{1}{2}\ln\left\lvert\dfrac{x+1}{x+3}\right\rvert + C$

Solution — Question 2

Decompose	$\dfrac{3x+1}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$	M1
Find A: $x=1$	$4 = A(3) \;\Rightarrow\; A = \frac{4}{3}$	A1
Find B: $x=-2$	$-5 = B(-3) \;\Rightarrow\; B = \frac{5}{3}$	A1
Integrate	$\frac{4}{3}\ln\lvert x-1\rvert + \frac{5}{3}\ln\lvert x+2\rvert + C$	M1

Final answer: $\dfrac{4}{3}\ln\lvert x-1\rvert + \dfrac{5}{3}\ln\lvert x+2\rvert + C$

Solution — Question 3

Decompose	$\dfrac{5x-2}{(x+1)(x-2)(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{x+3}$	M1
Find A: $x=-1$	$-7 = A(-3)(2) \;\Rightarrow\; A = \frac{7}{6}$	A1
Find B: $x=2$	$8 = B(3)(5) \;\Rightarrow\; B = \frac{8}{15}$	A1
Find C: $x=-3$	$-17 = C(-2)(-1) \;\Rightarrow\; C = -\frac{17}{2}$	A1
Integrate	$\frac{7}{6}\ln\lvert x+1\rvert + \frac{8}{15}\ln\lvert x-2\rvert - \frac{17}{2}\ln\lvert x+3\rvert + C$	M1

Final answer: $\dfrac{7}{6}\ln|x+1| + \dfrac{8}{15}\ln|x-2| - \dfrac{17}{2}\ln|x+3| + C$

Solution — Question 4

Decompose	$\dfrac{4x}{(x+1)^2(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-1}$	**M1**
Find C: $x = 1$	$4 = C(4) \Rightarrow C = 1$	**A1**
Find B: $x = -1$	$-4 = B(-2) \Rightarrow B = 2$	**A1**
Find A: expand & compare x^2	$0 = A + C \Rightarrow A = -1$	**A1**
Integrate	$-\ln|x+1| - \dfrac{2}{x+1} + \ln|x-1| + C$	**M1**

Final answer: $\ln\left|\dfrac{x-1}{x+1}\right| - \dfrac{2}{x+1} + C$

Solution — Question 5

From Q1	$A = \frac{1}{2},\ B = -\frac{1}{2}$	**R1**
Write definite form	$\left[\frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3|\right]_0^2$	**M1**
Apply log law	$\left[\frac{1}{2}\ln\left|\dfrac{x+1}{x+3}\right|\right]_0^2$	**M1**
Substitute limits	$\frac{1}{2}\ln\dfrac{3}{5} - \frac{1}{2}\ln\dfrac{1}{3}$	**A1**
Simplify	$\frac{1}{2}\ln\dfrac{9}{5}$	**A1**

Final answer: $\dfrac{1}{2}\ln\dfrac{9}{5}$

Solution — Question 6

Decompose	$\dfrac{x^2+2}{(x^2+1)(x+1)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x+1}$	**M1**
Find C: $x = -1$	$3 = C(2) \Rightarrow C = \frac{3}{2}$	**A1**
Compare x^2: $1 = A + C$	$A = 1 - \frac{3}{2} = -\frac{1}{2}$	**A1**
Compare constants: $2 = B + C$	$B = 2 - \frac{3}{2} = \frac{1}{2}$	**A1**
Integrate	$-\frac{1}{4}\ln(x^2+1) + \frac{1}{2}\arctan x + \frac{3}{2}\ln|x+1| + C$	**M1**

Final answer: $-\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\arctan x+\frac{3}{2}\ln|x+1|+C$

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