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Functions

Mistake Analysis – Set III

Course	IB Mathematics: Analysis & Approaches SL
Topic	Topic 2 – Functions: Exponential, Piecewise, Inverse & Modelling
Level	Hard (Paper 1 and Paper 2 style)
Questions	6
Marks	37 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Exponential functions: $f(x) = a \cdot b^x$. Evaluate by substitution; solve $b^x = c$ by matching bases or logarithms.

Piecewise functions: choose the correct rule based on which interval the input falls in.

Composite equations: substitute the inner function, then solve the resulting equation.

Quadratic models: the vertex gives the maximum or minimum; the roots give where the model equals zero.

Question 1 — Hard – Paper 1 — [6 marks]

Let $f(x) = 3 \cdot 2^x$.

(a) Find $f(0)$ and $f(3)$.

(b) Solve $f(x) = 96$.

MISTAKE ANALYSIS

(a) $f(0) = 3 \cdot 2^0 = 3 \cdot 1 = 3$. $f(3) = 3 \cdot 2^3 = 3 \cdot 8 = 24$. (b) $3 \cdot 2^x = 96 \Rightarrow 2^x = 32 = 2^5 \Rightarrow x = 5$. Students who compute $f(0) = 3 \cdot 0 = 0$ treat 2^0 as 0. Any nonzero base to the power 0 is 1, so $2^0 = 1$ and $f(0) = 3$. For part (b), students who write $2^x = 96/3 = 32$ but then guess $x = 32$ forget to express 32 as a power of 2: $32 = 2^5$, so $x = 5$.

Question 2 Hard – Paper 1 [6 marks]

A function is defined piecewise by

$$f(x) = \begin{cases} x^2 & x < 2 \\ 3x - 1 & x \geq 2 \end{cases}$$

Find $f(1)$, $f(2)$, and $f(-3)$.

MISTAKE ANALYSIS

$f(1)$: since $1 < 2$, use x^2: $f(1) = 1$. $f(2)$: since $2 \geq 2$, use $3x - 1$: $f(2) = 3(2) - 1 = 5$. $f(-3)$: since $-3 < 2$, use x^2: $f(-3) = (-3)^2 = 9$. Students who use x^2 for $f(2)$ pick the wrong piece – the condition $x \geq 2$ includes $x = 2$, so the second rule applies. Read the inequality boundaries carefully: $\geq$ includes the endpoint. Students who compute $f(-3) = -9$ forget that $(-3)^2 = 9$, not -9. Squaring a negative gives a positive.

Question 3 Hard – Paper 1 [6 marks]

Let $f(x) = x + 3$ and $g(x) = x^2$. Solve $(g \circ f)(x) = 25$.

MISTAKE ANALYSIS

$(g \circ f)(x) = g(f(x)) = g(x+3) = (x+3)^2$. $(x+3)^2 = 25 \Rightarrow x + 3 = \pm 5 \Rightarrow x = 2$ or $x = -8$. Students who take only the positive square root ($x + 3 = 5$) find $x = 2$ but miss $x = -8$. The equation $(x+3)^2 = 25$ has two solutions because both $+5$ and -5 square to 25. Students who compute $(f \circ g)(x) = f(g(x)) = x^2 + 3$ instead solve the wrong composition. $(g \circ f)$ applies f first.

Question 4 Hard – Paper 1 [6 marks]

The function f is defined by $f(x) = 2^x + 1$. Find $f^{-1}(x)$ and state its domain.

MISTAKE ANALYSIS

Let $y = 2^x + 1$. Swap: $x = 2^y + 1 \Rightarrow 2^y = x - 1 \Rightarrow y = \log_2(x - 1)$. $f^{-1}(x) = \log_2(x - 1)$. Domain: $x - 1 > 0 \Rightarrow x > 1$. The domain of f^{-1} is the range of f. Since $2^x > 0$, $f(x) = 2^x + 1 > 1$, so the range of f is $(1, \infty)$, which becomes the domain of f^{-1}. Students who write $f^{-1}(x) = \log_2 x - 1$ misplace the *bracket. The inverse is $\log_2(x-1)$, with the subtraction inside the logarithm.*

Question 5 Hard – Paper 1 [6 marks]

The function f is defined by $f(x) = \dfrac{x+1}{x-1}$, $x \neq 1$. Show that $f(f(x)) = x$, and state what this tells you about f.

MISTAKE ANALYSIS

$f(f(x)) = \dfrac{f(x)+1}{f(x)-1} = \dfrac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$. Numerator: $\dfrac{x+1+(x-1)}{x-1} = \dfrac{2x}{x-1}$. Denominator: $\dfrac{x+1-(x-1)}{x-1} = \dfrac{2}{x-1}$. $f(f(x)) = \dfrac{2x/(x-1)}{2/(x-1)} = \dfrac{2x}{2} = x$. ✓ This shows f is self-inverse: $f^{-1} = f$. Students who try to simplify without a common denominator make algebraic errors. Combine the terms in numerator and denominator over the common denominator $(x-1)$ first, then divide.

Question 6 Hard – Paper 2 [7 marks]

A ball is thrown and its height in metres after t seconds is $h(t) = -5t^2 + 20t + 1$.

(a) Find the maximum height and the time at which it occurs.

(b) Find the time at which the ball hits the ground, to 3 significant figures.

MISTAKE ANALYSIS

(a) Maximum at vertex: $t = \dfrac{-b}{2a} = \dfrac{-20}{2(-5)} = 2$ seconds. $h(2) = -5(4)+20(2)+1 = -20+40+1 = 21$ m. (b) Ground: $h(t) = 0$: $-5t^2+20t+1 = 0$. Using the quadratic formula: $t = \dfrac{-20 \pm \sqrt{400+20}}{-10} = \dfrac{-20 \pm \sqrt{420}}{-10}$. $t \approx 4.05$ s (taking the positive root; reject $t \approx -0.05$). Students who give the maximum height as $t = 2$ (the time) instead of $h(2) = 21$ (the height) confuse the two. The question asks for the height; substitute $t = 2$ into h. Students who accept the negative root $t \approx -0.05$ ignore that time cannot be negative in this context.

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

(a) $f(0) = 3 \cdot 2^0 = 3$; $f(3) = 3 \cdot 8$	$f(0) = 3,\ f(3) = 24$	**M1 A1**
(b) $2^x = 32 = 2^5$	$x = 5$	**M1 A1**

Solution – Question 2

$f(1) = 1^2$ $(1 < 2)$; $f(2) = 3(2) - 1$ $(2 \geq 2)$; $f(-3) = (-3)^2$	$f(1) = 1,\ f(2) = 5,\ f(-3) = 9$	**M1 A1 A1**

Solution – Question 3

$(x+3)^2 = 25 \Rightarrow x + 3 = \pm 5$	$x = 2$ or $x = -8$	**M1 A1**

Solution – Question 4

$2^y = x - 1 \Rightarrow y = \log_2(x-1)$	$f^{-1}(x) = \log_2(x-1)$	**M1 A1**
Domain: range of f	$x > 1$	**R1**

Solution – Question 5

Num $= \frac{2x}{x-1}$; Den $= \frac{2}{x-1}$; ratio $= x$	$f(f(x)) = x$	**M1 A1**
f is self-inverse	$f^{-1} = f$	**R1**

Solution – Question 6

(a) $t = \frac{-20}{2(-5)} = 2$; $h(2) = 21$	21 m at $t = 2$ s	**M1 A1**
(b) $-5t^2 + 20t + 1 = 0$; reject negative root	$t \approx 4.05$ s	**M1 A1**
