Functions

Mistake Analysis – Set I

Course	IB Mathematics: Analysis & Approaches SL
Topic	Topic 2 – Functions
Level	Medium → Hard (Paper 1 and Paper 2 style)
Questions	6
Marks	34 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Composite functions: $(f \circ g)(x) = f(g(x))$ means apply g first, then f. Order matters.

Inverse functions: swap x and y, then solve for y. The domain of f^{-1} is the range of f.

Domain restrictions: square roots require a non-negative argument; denominators cannot be zero.

Transformations: $y = af(x-h)+k$ stretches by a, shifts right by h, shifts up by k.

Question 1

Medium – Paper 1 [6 marks]

Let $f(x) = 2x + 1$ and $g(x) = x^2 - 3$.

(a) Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

(b) Show that $(f \circ g)(x) \neq (g \circ f)(x)$ in general.

MISTAKE ANALYSIS

(a) $(f \circ g)(x) = f(g(x)) = 2(x^2-3)+1 = 2x^2-5$. $(g \circ f)(x) = g(f(x)) = (2x+1)^2-3 = 4x^2+4x+1-3 = 4x^2+4x-2$. (b) $2x^2-5 \neq 4x^2+4x-2$ for general x (e.g. at $x=0$: $-5 \neq -2$). Students who compute $(f \circ g)(x)$ as $g(f(x))$ reverse the order. In $f \circ g$, the function g is applied first (it is closest to x), then f. The composition $f \circ g$ means "f of g of x". Students who write $(f \circ g)(x) = f(x) \cdot g(x)$ confuse composition with multiplication. Composition substitutes one function into another; it is not a product.

Question 2 Hard – Paper 1 [6 marks]

The function f is defined by $f(x) = \dfrac{3x-2}{x+1}$, $x \neq -1$. Find $f^{-1}(x)$ and state its domain.

MISTAKE ANALYSIS

Let $y = \dfrac{3x-2}{x+1}$. Swap x and y: $x = \dfrac{3y-2}{y+1}$. $x(y+1) = 3y-2 \Rightarrow xy + x = 3y - 2 \Rightarrow xy - 3y = -2 - x \Rightarrow y(x-3) = -(x+2) \Rightarrow y = \dfrac{x+2}{3-x}$. $f^{-1}(x) = \dfrac{x+2}{3-x}$. Domain: $x \neq 3$ (the range of f excludes 3). Students who "invert" by writing $f^{-1}(x) = \dfrac{x+1}{3x-2}$ (taking the reciprocal) confuse the inverse function with the reciprocal. The inverse is found by swapping variables and solving, not by flipping the fraction. The domain of f^{-1} is the range of f. As $x \to \infty$, $f(x) \to 3$, so 3 is excluded.

Question 3 Medium – Paper 1 [5 marks]

Find the largest possible domain of the function $f(x) = \dfrac{\sqrt{x-2}}{x-5}$.

MISTAKE ANALYSIS

Two conditions: the square root requires $x - 2 \geq 0 \Rightarrow x \geq 2$; the denominator requires $x - 5 \neq 0 \Rightarrow x \neq 5$. Domain: $x \geq 2$ and $x \neq 5$, i.e. $[2, 5) \cup (5, \infty)$. Students who only consider the square root ($x \geq 2$) forget the denominator restriction at $x = 5$. Students who write $x > 2$ (strict) exclude $x = 2$, but $\sqrt{0} = 0$ is defined, so $x = 2$ is included. The square root requires $\geq$, not $>$.

Question 4 Medium – Paper 1 [5 marks]

Find the range of the function $f(x) = x^2 - 4x + 7$.

MISTAKE ANALYSIS

Complete the square: $f(x) = (x-2)^2 - 4 + 7 = (x-2)^2 + 3$. The minimum value is 3, at $x = 2$. Since the parabola opens upward, the range is $f(x) \geq 3$, i.e. $[3, \infty)$. Students who state the range as $f(x) \geq 0$ assume all quadratics have minimum 0 – only $y = x^2$ does. Here the vertex is at $y = 3$. Students who give the range as the domain (all real x) confuse domain and range. The domain is $\mathbb{R}$; the range is $[3, \infty)$.

Question 5

Medium – Paper 1 [6 marks]

The graph of $y = f(x)$ is transformed to $y = 2f(x-3)+1$. Describe fully the sequence of transformations, in the correct order.

MISTAKE ANALYSIS

Three transformations: (1) horizontal translation 3 units right (from $x-3$); (2) vertical stretch by scale factor 2 (from the coefficient 2); (3) vertical translation 1 unit up (from $+1$). A point (a, b) on $y = f(x)$ maps to $(a+3, 2b+1)$. Students who describe $x-3$ as a shift left confuse the direction: replacing x with $x-3$ shifts the graph right, not left. Students who apply the $+1$ before the stretch get the wrong image: the order for vertical transformations is stretch first, then translate. $2f(x-3)+1$ stretches by 2, then adds 1.

Question 6

Hard – Paper 1 [6 marks]

Solve $|2x-3| = 5$, and hence state the solutions of $|2x-3| < 5$.

MISTAKE ANALYSIS

$|2x-3| = 5$ gives two cases: $2x-3 = 5 \Rightarrow x = 4$; or $2x-3 = -5 \Rightarrow x = -1$. For $|2x-3| < 5$: this is $-5 < 2x-3 < 5 \Rightarrow -2 < 2x < 8 \Rightarrow -1 < x < 4$. Students who solve only $2x-3 = 5$ and report $x = 4$ miss the negative case. The absolute value equation $|A| = 5$ has two solutions: $A = 5$ and $A = -5$. For the inequality, students who write $2x-3 < 5$ only (dropping the lower bound) get $x < 4$ but miss $x > -1$. The inequality $|A| < 5$ means $-5 < A < 5$, a double inequality.

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

(a) $f(g(x)) = 2(x^2-3)+1$	$2x^2-5$	**M1 A1**
$g(f(x)) = (2x+1)^2-3$	$4x^2+4x-2$	**A1**
(b) At $x=0$: $-5 \neq -2$, so not equal		**R1**

Solution – Question 2

Swap and solve: $x(y+1)=3y-2 \Rightarrow y(x-3) = -(x+2)$	$f^{-1}(x) = \dfrac{x+2}{3-x}$	**M1 A1**
Domain: range of f excludes 3	$x \neq 3$	**R1**

Solution – Question 3

$x-2 \geq 0$ and $x-5 \neq 0$	$[2,5) \cup (5,\infty)$	**M1 A1**

Solution – Question 4

$(x-2)^2+3$; minimum 3 at $x=2$	$[3,\infty)$	**M1 A1**

Solution – Question 5

Right 3; vertical stretch factor 2; up 1		**M1 A1**
Point $(a,b) \to (a+3,\ 2b+1)$		**A1**

Solution – Question 6

$2x - 3 = 5$ or $x = 4$ or $x = -1$ **M1 A1**
$2x - 3 = -5$
$|2x - 3| < 5 \Rightarrow -1 < x < 4$ **A1**
$-5 < 2x - 3 < 5$