Exponents & Logarithms

Mistake Analysis – Set III

Course	IB Mathematics: Analysis & Approaches SL
Topic	Topic 1 – Number & Algebra: Exponents and Logarithms
Level	Medium → Hard (Paper 1 and Paper 2 style)
Questions	6
Marks	36 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Logarithm laws – Paper 1 toolkit: $\log \frac{A}{B} = \log A - \log B$; $\log(AB) = \log A + \log B$; $\log A^n = n \log A$.

Expressing in terms of given values: factorise the argument into known prime powers before applying laws.

Inverse relationship: $a^{\log_a x} = x$ and $\log_a(a^x) = x$. Use these to check solutions.

Two-variable compound interest: divide the two equations to eliminate P, then solve for the rate.

Question 1 Medium – Paper 1 [4 marks]

Without using a calculator, evaluate $\log_2 96 - \log_2 3 - \log_2 4$.

MISTAKE ANALYSIS

$\log_2 96 - \log_2 3 - \log_2 4 = \log_2\left(\frac{96}{3 \times 4}\right) = \log_2\left(\frac{96}{12}\right) = \log_2 8 = \log_2 2^3 = 3$. *Students who evaluate each term separately attempt* $\log_2 96$, *which requires recognising* $96 = 2^5 \times 3$, *giving* $\log_2 96 = 5 + \log_2 3$ *– correct but circular. The direct approach is to combine all three logs into one fraction using the quotient rule, then simplify. A common arithmetic error:* $96 \div 3 = 32$ *(correct) but then* $32 \div 4 = 8$ *(correct) – so this step is fine. The trap is forgetting that subtraction of logs corresponds to division, not subtraction, of arguments.*

Question 2 Hard – Paper 1 [5 marks]

Solve $\log_2(x+3) = 3 - \log_2 x$, giving your answer in the form $\dfrac{-a+\sqrt{b}}{2}$ where $a, b \in \mathbb{Z}^+$.

MISTAKE ANALYSIS

Rearrange: $\log_2(x+3) + \log_2 x = 3 \Rightarrow \log_2[x(x+3)] = 3 \Rightarrow x(x+3) = 8$. $x^2 + 3x - 8 = 0 \Rightarrow x = \dfrac{-3+\sqrt{9+32}}{2} = \dfrac{-3+\sqrt{41}}{2}$. *Reject* $x = \dfrac{-3-\sqrt{41}}{2}$ *since this is negative and the domain requires* $x > 0$. *Domain check:* $\log_2(x+3)$ *requires* $x > -3$ *and* $\log_2 x$ *requires* $x > 0$, *so* $x > 0$. *Students who move* $\log_2 x$ *to the left but do not add (only combine as quotient) write* $\log_2 \frac{x+3}{x} = 3$, *giving* $\frac{x+3}{x} = 8$, *so* $x + 3 = 8x$, $x = \frac{3}{7}$ *– wrong. Subtraction of logs is division; addition is multiplication. Here you are adding the logs, so multiply the arguments.*

Question 3 Medium – Paper 1 [5 marks]

Given that $\log_{10} 2 = p$ and $\log_{10} 3 = q$, express $\log_{10}\sqrt{24}$ in terms of p and q.

MISTAKE ANALYSIS

$\sqrt{24} = \sqrt{4 \times 6}$, *so* $\log_{10}\sqrt{24} = \frac{1}{2}\log_{10} 24 = \frac{1}{2}\log_{10}(8 \times 3) = \frac{1}{2}(3\log_{10} 2 + \log_{10} 3) = \frac{3p+q}{2}$. *Students who write* $\log_{10}\sqrt{24} = \log_{10} 24^{1/2} = \frac{1}{2} \times 24 \times \log_{10} \ldots$ *confuse the power rule with multiplication. The power rule:* $\log(A^n) = n\log A$. *Here* $n = \frac{1}{2}$ *and* $A = 24$: *result is* $\frac{1}{2}\log_{10} 24$, *not* $12\log_{10} 24$. *Also: some students express* $24 = 2^3 \times 3$ *but then write* $\log_{10}(2^3 \times 3) = 3p \times q$ *(multiplying instead of adding). The product rule gives* $\log(AB) = \log A + \log B$, *not* $\log A \times \log B$.

Question 4 Medium – Paper 2 [5 marks]

\$P is invested at an annual compound interest rate of r%. After 10 years the amount is \$1500. After 20 years the amount is \$2400. Find P and r, giving r correct to 2 decimal places.

MISTAKE ANALYSIS

$P(1+r/100)^{10} = 1500$ *...(1)* $P(1+r/100)^{20} = 2400$ *...(2) Divide (2) by (1):* $(1+r/100)^{10} = \frac{2400}{1500} = \frac{8}{5}$. $1+r/100 = \left(\frac{8}{5}\right)^{1/10}$, *so* $r = 100\left[\left(\frac{8}{5}\right)^{1/10} - 1\right] \approx 4.81\%$. *From (1):* $P = \frac{1500}{(8/5)} = \frac{1500\times5}{8} = \$937.50$. *Students who subtract equation (1) from (2) instead of dividing get* $P[(1+r/100)^{20} - (1+r/100)^{10}] = 900$, *which* *cannot be solved without additional information. Division is the key step – it eliminates P.*

Question 5 Hard – Paper 1 [7 marks]

Solve $2\log_5 x - \log_5(x-4) = 2$, stating any values you reject and why.

MISTAKE ANALYSIS

$\log_5(x^2) - \log_5(x-4) = 2 \Rightarrow \log_5\left(\frac{x^2}{x-4}\right) = 2 \Rightarrow \frac{x^2}{x-4} = 25$. $x^2 = 25(x-4) = 25x - 100 \Rightarrow x^2 - 25x + 100 = 0 \Rightarrow (x-5)(x-20) = 0$. $x = 5$ *or* $x = 20$. *Domain:* $\log_5(x-4)$ *requires* $x > 4$*; both solutions satisfy* $x > 4$. *Verify* $x = 5$*:* $2\log_5 5 - \log_5 1 = 2(1) - 0 = 2\checkmark$. *Verify* $x = 20$*:* $\log_5\left(\frac{400}{16}\right) = \log_5 25 = 2\checkmark$. *Both accepted. Students who check the domain only for* $\log_5(x-4)$ *and not for* $\log_5 x$ *are correct here (since* $x > 4$ *implies* $x > 0$*) but should state both conditions. Students who reject* $x = 5$ *because* $5 - 4 = 1$ *and* $\log_5 1 = 0$ *make an error – a zero argument to one log is fine; it is a negative or zero argument that is undefined.*

Question 6 Hard – Paper 1 [10 marks]

(a) Show that $\log_a b \cdot \log_b c = \log_a c$.

(b) Hence find the value of $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8$.

(c) State the general result for a product of the form $\log_a b \cdot \log_b c \cdots \log_y z$.

MISTAKE ANALYSIS

(a) $\log_a b \cdot \log_b c = \dfrac{\ln b}{\ln a} \cdot \dfrac{\ln c}{\ln b} = \dfrac{\ln c}{\ln a} = \log_a c$. $\checkmark$ *(b) The product telescopes:* $\log_2 3 \cdot \log_3 4 \cdots \log_7 8 = \log_2 8 = \log_2 2^3 = 3$. *(c)* $\log_a b \cdot \log_b c \cdots \log_y z = \log_a z$ *(all intermediate bases cancel). Students who attempt to evaluate* $\log_2 3$ *numerically and multiply through numerically arrive at the right answer but do not demonstrate the telescoping – earning no method marks. The key insight: consecutive logs form a chain where the "output" base of each factor equals the "input" base of the next. Change of base reveals the cancellation immediately.*

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

$\log_2\left(\frac{96}{3\times 4}\right) = 3$		**M1 A1**
$\log_2\left(\frac{96}{12}\right) =$		
$\log_2 8 = \log_2 2^3$		

Solution – Question 2

$\log_2[x(x+3)] = 3 \Rightarrow x^2 + 3x - 8 = 0$; reject $x = \frac{-3-\sqrt{41}}{2} < 0$	$\frac{-3+\sqrt{41}}{2}$	**M1 A1 R1**

Solution – Question 3

$\frac{1}{2}\log_{10}(8\times 3) = \frac{1}{2}(3p+q)$	$\frac{3p+q}{2}$	**M1 A1**

Solution – Question 4

Divide: $(1 + r/100)^{10} = 8/5$; $r = 100[(8/5)^{1/10} - 1]$	$r \approx 4.81\%$	**M1 A1**
$P = 1500/(8/5)$	\$937.50	**A1**

Solution – Question 5

$\log_5(x^2/(x-4)) = 2 \Rightarrow x^2 - 25x + 100 = (x-5)(x-20) = 0$	$x = 5$ or $x = 20$	**M1 A1**
Domain $x > 4$: both accepted; verify $x = 5$: $2 - 0 = 2$ ✓; $x = 20$: $\log_5 25 = 2$ ✓		**R1**

Solution – Question 6

(a) $\frac{\ln b}{\ln a} \cdot \frac{\ln c}{\ln b} = \frac{\ln c}{\ln a} = \log_a c$ ✓		**M1 A1**
(b) Product telescopes: $\log_2 3 \cdot \log_3 4 \cdots \log_7 8 = \log_2 8 = 3$	3	**M1 A1**
(c) $\log_a b \cdot \log_b c \cdots \log_y z = \log_a z$		**A1**

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