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Exponents & Logarithms

Mistake Analysis – Set II

Course	IB Mathematics: Analysis & Approaches SL
Topic	Topic 1 – Number & Algebra: Exponents and Logarithms
Level	Medium → Hard (Paper 1 and Paper 2 style)
Questions	6
Marks	35 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Quadratic substitution: If a^{2x} appears with a^x, let $u = a^x$ and solve the resulting quadratic in u.

Domain of logarithms: $\log_a f(x)$ is defined only when $f(x) > 0$. Check all solutions.

Change of base: $\log_b a = \dfrac{\ln a}{\ln b}$; so $\log_a x \cdot \log_x a = 1$.

Exponential models: $P = P_0\, e^{kt}$. Solve for t by isolating the exponential, then taking ln of both sides.

Question 1

Medium – Paper 1 [5 marks]

Solve $3^{2x} - 4 \cdot 3^x + 3 = 0$, giving your answers as integers.

MISTAKE ANALYSIS

Let $u = 3^x$: $u^2 - 4u + 3 = (u-1)(u-3) = 0$, so $u = 1$ or $u = 3$. $3^x = 1 = 3^0 \Rightarrow x = 0$; $3^x = 3 = 3^1 \Rightarrow x = 1$. The key recognition: $3^{2x} = (3^x)^2$, not $2 \cdot 3^x$. Once you write $u = 3^x$, the equation is a standard quadratic. Students who attempt to take $\log_3$ of both sides directly encounter $\log_3(3^{2x} - 4 \cdot 3^x + 3)$, which cannot be simplified – logarithms do not distribute over subtraction. The substitution $u = 3^x$ is the only clean approach.

Question 2

Hard – Paper 1 [6 marks]

The function $f(x) = 2\log_3(x-1)+1$ is defined for $x > 1$.

(a) State the equation of the vertical asymptote and the range of f.

(b) Find the x-intercept, giving your answer in the form $\dfrac{a+\sqrt{b}}{c}$ where $a, b, c \in \mathbb{Z}^+$.

MISTAKE ANALYSIS

(a) Asymptote: $x = 1$. *Range:* $f(x) \in \mathbb{R}$ *(all real numbers). (b) Set* $f(x) = 0$: $2\log_3(x-1) = -1 \Rightarrow \log_3(x-1) = -\frac{1}{2} \Rightarrow x - 1 = 3^{-1/2} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. $x = 1 + \dfrac{\sqrt{3}}{3} = \dfrac{3+\sqrt{3}}{3}$. *Students who write* $3^{-1/2} = \frac{1}{3}$ *(halving the base instead of taking the reciprocal of the square root) get* $x - 1 = \frac{1}{3}$, *giving* $x = \frac{4}{3}$ *– wrong.* $3^{-1/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. *Students who forget the vertical shift of 1 and set* $2\log_3(x-1) = 0$ *rather than* -1 *find* $x = 2$ *– this locates a zero of the unshifted function. Always set the full expression equal to zero.*

Question 3 Medium – Paper 1 [5 marks]

Prove that $\log_a x = \dfrac{1}{\log_x a}$ for $a, x > 0$, $a, x \neq 1$.

MISTAKE ANALYSIS

$\log_a x = \dfrac{\ln x}{\ln a}$; $\log_x a = \dfrac{\ln a}{\ln x}$; *so* $\dfrac{1}{\log_x a} = \dfrac{\ln x}{\ln a} = \log_a x$. ✓ *Students who attempt to "prove" this by substituting a numerical example (e.g.* $a = 2$, $x = 8$*) demonstrate a specific case, not a general proof. A proof requires algebraic argument valid for all permissible values of* a *and* x.

Question 4 Hard – Paper 1 [6 marks]

Solve $\ln(x+2) + \ln(x-1) = \ln(4x-2)$, stating any values you reject and why.

MISTAKE ANALYSIS

Combine left side: $\ln[(x+2)(x-1)] = \ln(4x-2)$. $(x+2)(x-1) = 4x-2 \Rightarrow x^2 + x - 2 = 4x - 2 \Rightarrow x^2 - 3x = 0 \Rightarrow x(x-3) = 0$. $x = 0$ *or* $x = 3$. *Domain:* $\ln(x-1)$ *requires* $x > 1$*; so reject* $x = 0$. *Accept* $x = 3$. *Verify:* $\ln 5 + \ln 2 = \ln 10$ *and* $\ln(4(3)-2) = \ln 10$. ✓ *Students who combine logs correctly but forget to check the domain accept* $x = 0$, *which makes* $\ln(x-1) = \ln(-1)$ *undefined. Every solution to a logarithmic equation must be substituted back to verify all arguments are positive.*

Question 5 Hard – Paper 2 [6 marks]

A population grows according to the model $P(t) = 2000\,e^{0.03t}$, where t is time in years.

(a) Find the population when $t = 0$ and interpret this value.

(b) Find the time at which the population first reaches 5000, giving your answer to 3 significant figures.

MISTAKE ANALYSIS

(a) $P(0) = 2000\,e^0 = 2000$. *This is the initial population. (b)* $2000\,e^{0.03t} = 5000 \Rightarrow e^{0.03t} = 2.5 \Rightarrow 0.03t = \ln 2.5 \Rightarrow t = \dfrac{\ln 2.5}{0.03} \approx 30.5$ *years. Students who write* $0.03t = \ln 2.5 - \ln 2000$ *have taken* $\ln$ *of the right side but subtracted* $\ln 2000$ *incorrectly. The correct step is to divide both sides by 2000 first, giving* $e^{0.03t} = 2.5$, *before taking* $\ln$. *Students who give* $t \approx 30.543$ *to 5 s.f., rather than* $t \approx 30.5$ *to 3 s.f., losing the A1 mark. Read the required precision carefully.*

Question 6 Hard – Paper 1 [7 marks]

Solve the simultaneous equations $2^x \cdot 3^y = 72$ and $2^x - 3^y = -1$.

MISTAKE ANALYSIS

Let $u = 2^x$ *and* $v = 3^y$: $uv = 72$ *and* $u - v = -1$, *so* $u = v - 1$. $(v-1)v = 72 \Rightarrow v^2 - v - 72 = 0 \Rightarrow (v-9)(v+8) = 0$. $v = 9$ *(reject* $v = -8$, *since* $3^y > 0$ *always) or* $v = 9$: $3^y = 9 = 3^2 \Rightarrow y = 2$. $u = 9 - 1 = 8 = 2^3 \Rightarrow x = 3$. *Solution:* $x = 3$, $y = 2$. *Verify:* $2^3 \cdot 3^2 = 8 \cdot 9 = 72\,\checkmark$; $8 - 9 = -1\,\checkmark$. *Students who take* $\log$ *of both original equations and try to solve the resulting linear system make an error on the second equation:* $\log(2^x - 3^y) \neq \log(2^x) - \log(3^y)$. *Logarithms do not distribute over subtraction. The substitution* $u = 2^x$, $v = 3^y$ *is the correct approach.*

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

$u = 3^x$: $u^2 - 4u + 3 = (u-1)(u-3) = 0$; $u = 1 \Rightarrow x = 0$; $u = 3 \Rightarrow x = 1$ — $x = 0$ or $x = 1$ — **M1 A1**

Solution – Question 2

(a) Asymptote $x = 1$; range $f(x) \in \mathbb{R}$ — **A1**

(b) $\log_3(x-1) = -\frac{1}{2} \Rightarrow x - 1 = 3^{-1/2} = \frac{\sqrt{3}}{3}$ — $\dfrac{3+\sqrt{3}}{3}$ — **M1 A1**

Solution – Question 3

$\log_a x = \frac{\ln x}{\ln a}$; $\frac{1}{\log_x a} = \frac{1}{\ln a / \ln x} = \frac{\ln x}{\ln a} = \log_a x$ — ✓ — **M1 A1**

Solution – Question 4

$\ln[(x+2)(x-1)] = \ln(4x-2) \Rightarrow x^2 - 3x = 0 \Rightarrow x = 0$ or $x = 3$ — **M1**

Reject $x = 0$: domain requires $x > 1$ ✓ — $x = 3$ — **R1 A1**

Solution – Question 5

(a) $P(0) = 2000$: initial population		**A1**
(b) $e^{0.03t} = 2.5 \Rightarrow t = \frac{\ln 2.5}{0.03}$	$t \approx 30.5$ years	**M1 A1**

Solution – Question 6

$u = 2^x,\ v = 3^y$: substitute $u = v - 1$ into $uv = 72$	$(v-1)v = 72$	**M1**
$v^2 - v - 72 = 0 \Rightarrow (v-9)(v+8) = 0$; reject $v = -8$ since $3^y > 0$	$v = 9$	**M1 A1 R1**
$3^y = 9 \Rightarrow y = 2$; $2^x = v - 1 = 8 \Rightarrow x = 3$	$x = 3,\ y = 2$	**A1**
