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Exponents & Logarithms

Mistake Analysis – Set I

Course	IB Mathematics: Analysis & Approaches SL
Topic	Topic 1 – Number & Algebra: Exponents and Logarithms
Level	Medium → Hard (Paper 1 and Paper 2 style)
Questions	6
Marks	34 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Laws of exponents: $a^m \cdot a^n = a^{m+n}$; $(a^m)^n = a^{mn}$; $a^0 = 1$; $a^{-n} = \frac{1}{a^n}$.

Laws of logarithms: $\log(AB) = \log A + \log B$; $\log\left(\frac{A}{B}\right) = \log A - \log B$; $\log A^n = n \log A$.

Change of base: $\log_b a = \dfrac{\ln a}{\ln b} = \dfrac{\log a}{\log b}$.

Key link: $\log_a b = c \Leftrightarrow a^c = b$. The logarithm is the exponent.

Question 1

Medium – Paper 1 [5 marks]

Solve $2^{x+1} = 32$, giving your answer as an integer.

MISTAKE ANALYSIS

Write both sides as powers of 2: $2^{x+1} = 2^5$, so $x + 1 = 5$, giving $x = 4$. Students who take $\log_2$ of both sides reach $x + 1 = \log_2 32 = 5$, the same result – valid but slower on Paper 1. The most common error: writing $2^{x+1} = 2 \cdot 2^x$ and then $2 \cdot 2^x = 32 \Rightarrow 2^x = 16 = 2^4 \Rightarrow x = 4$ – this is correct but relies on recognising $16 = 2^4$. A critical error: solving $2(x + 1) = 32 \Rightarrow x + 1 = 16 \Rightarrow x = 15$. This treats the exponent as a multiplier. The base 2 is raised to the power $x + 1$; it is not multiplied by $x + 1$.

Question 2

Hard – Paper 1 [6 marks]

Solve $\log_3 x + \log_3(x - 6) = 3$, stating any values you reject and why.

MISTAKE ANALYSIS

Combine: $\log_3[x(x-6)] = 3 \Rightarrow x(x-6) = 3^3 = 27$. $x^2 - 6x - 27 = 0 \Rightarrow (x-9)(x+3) = 0 \Rightarrow x = 9$ *or* $x = -3$. *Reject* $x = -3$*:* $\log_3(-3)$ *is undefined (logarithm of a negative number does not exist in* $\mathbb{R}$*). Accept* $x = 9$*. Verify:* $\log_3 9 + \log_3 3 = 2 + 1 = 3\checkmark$*. Students who solve the quadratic correctly but accept both solutions lose the R1 mark. You must check every solution in the original equation – logarithm equations frequently produce extraneous roots. The domain of* $\log_3 x$ *is* $x > 0$*, and* $\log_3(x-6)$ *requires* $x > 6$*. So the domain is* $x > 6$*, which immediately excludes* $x = -3$*.*

Question 3 Medium – Paper 1 [5 marks]

Express $\log_a\left(\frac{x^2\sqrt{y}}{z^3}\right)$ in terms of $\log_a x$, $\log_a y$, and $\log_a z$.

MISTAKE ANALYSIS

$\log_a\left(\frac{x^2\sqrt{y}}{z^3}\right) = \log_a(x^2) + \log_a(y^{1/2}) - \log_a(z^3) = 2\log_a x + \frac{1}{2}\log_a y - 3\log_a z$. *Students who write* $\log_a(\sqrt{y}) = 2\log_a y$ *(doubling instead of halving) misapply the power rule.* $\sqrt{y} = y^{1/2}$*, so* $\log_a(y^{1/2}) = \frac{1}{2}\log_a y$*. Also: students who treat division as subtraction of the entire denominator write* $-\log_a(z^3) = -\log_a z$ *(forgetting the power 3). The power rule* $\log_a(z^3) = 3\log_a z$ *must be applied before the sign is attached.*

Question 4 Hard – Paper 1 [6 marks]

Solve $e^{2x} - 5e^x + 6 = 0$, giving your answers in the form $\ln k$ where $k \in \mathbb{Z}^+$.

MISTAKE ANALYSIS

Let $u = e^x$*: the equation becomes* $u^2 - 5u + 6 = 0$*, which factorises as* $(u-2)(u-3) = 0$*. The substitution* $u = e^x$ *is mandatory – this is a quadratic in* e^x*, not in* x*. A common error: writing* $e^{2x} = 2e^x$ *(confusing the exponent rule with differentiation, where* $\frac{d}{dx}[e^{2x}] = 2e^{2x}$*). This gives* $2e^x - 5e^x + 6 = -3e^x + 6 = 0$*, so* $e^x = 2$ *– which happens to produce one correct solution but misses* $x = \ln 3$ *entirely. The safe route: always substitute* $u = e^x$ *first, solve the quadratic in* u*, then convert back.*

Question 5

Hard – Paper 2 [6 marks]

\$5 000 is invested in an account paying 3.5% compound interest per annum. Find the minimum number of complete years for the investment to exceed \$7 000.

MISTAKE ANALYSIS

$5000 \times (1.035)^n > 7000 \Rightarrow 1.035^n > 1.4$. Take logarithms: $n\ln(1.035) > \ln(1.4) \Rightarrow n > \dfrac{\ln 1.4}{\ln 1.035} \approx \dfrac{0.3365}{0.0344} \approx 9.78$. Since n must be a whole number of complete years: $n = 10$. Verify: $5000 \times (1.035)^{10} \approx \$7053 > \$7000$ ✓. $5000 \times (1.035)^9 \approx \$6814 < \$7000$ ✓. Students who round 9.78 down to $n = 9$ find the investment has not yet reached \$7 000. The question asks for the minimum number of complete years for the amount to exceed \$7 000, so round up to $n = 10$. Students who use simple interest ($5000 \times 0.035 \times n > 2000$) get $n > 11.4$, giving a wrong answer of 12 years.

Question 6

Hard – Paper 1 [6 marks]

(a) Show that $\log_4 8 = \dfrac{3}{2}$.

(b) Hence solve $\log_4 x + \log_4(x-2) = \dfrac{3}{2}$, stating any values you reject.

MISTAKE ANALYSIS

(a) $\log_4 8 = \dfrac{\ln 8}{\ln 4} = \dfrac{3\ln 2}{2\ln 2} = \dfrac{3}{2}$. ✓ Alternatively: $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$. ✓ (b) $\log_4[x(x-2)] = \frac{3}{2} = \log_4 8 \Rightarrow x(x-2) = 8 \Rightarrow x^2 - 2x - 8 = 0 \Rightarrow (x-4)(x+2) = 0$. $x = 4$ or $x = -2$. Reject $x = -2$ (domain requires $x > 0$ and $x > 2$, so $x > 2$). Accept $x = 4$. Verify: $\log_4 4 + \log_4 2 = 1 + \frac{1}{2} = \frac{3}{2}$. ✓ The word "hence" requires use of part (a). Students who solve part (b) from scratch using change of base, without referencing the result $\log_4 8 = 3/2$, do not earn the M1 "hence" mark. The connection is: set the combined log equal to $\log_4 8$ to convert directly to $x(x-2) = 8$.

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

$2^{x+1} = 2^5$; equate exponents	$x + 1 = 5 \Rightarrow x = 4$	**M1**

Solution – Question 2

$\log_3[x(x-6)] = 3 \Rightarrow x^2 - 6x - 27 = 0 \Rightarrow (x-9)(x+3) = 0$	$x = 9$ or $x = -3$	**M1**
Reject $x = -3$: logarithm undefined for negative argument; domain requires $x > 6$	$x = 9$	**R1**

Solution – Question 3

$\log_a(x^2) + \log_a(y^{1/2}) - \log_a(z^3)$	$2\log_a x + \frac{1}{2}\log_a y - 3\log_a z$	**M1**

Solution – Question 4

$u = e^x$: $u^2 - 5u + 6 = (u-2)(u-3) = 0$; $u = 2$ or $u = 3$	$x = \ln 2$ or $x = \ln 3$	**M1**

Solution – Question 5

$1.035^n > 1.4$; $n > \frac{\ln 1.4}{\ln 1.035} \approx \frac{0.3365}{0.0344} \approx 9.78$; round up $\quad n = 10$ years **M1**

Verify: $5000 \times 1.035^{10} \approx 7053 > 7000$ ✓; $5000 \times 1.035^{9} \approx 6814 < 7000$ ✓ **R1**

Solution – Question 6

(a) $\log_4 8 = \frac{\ln 8}{\ln 4} = \frac{3 \ln 2}{2 \ln 2} = \frac{3}{2}$ ✓ (or: $4^{3/2} = (\sqrt{4})^3 = 8$ ✓) $\quad \frac{3}{2}$ **M1 A1**

(b) $\log_4[x(x-2)] = \log_4 8$ (uses part a)$\Rightarrow x(x-2) = 8$ **M1**

$x^2 - 2x - 8 = (x-4)(x+2) = 0 \Rightarrow x = 4$ or $x = -2$ $\quad x = 4$ **M1 A1**

Reject $x = -2$: $\log_4(x-2)$ requires $x > 2$, so $x = -2$ is outside the domain **R1**
