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Vector Equations of Lines

Recognition Training – Set I

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 3 – Geometry & Trigonometry
Level	Easy → Medium
Questions	6
Total marks	31
Instructions	Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

The vector equation of a line through point A with position vector $\mathbf{a}$ and direction vector $\mathbf{d}$ is:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \quad \lambda \in \mathbb{R}.$$

Two lines are **parallel** if their direction vectors are scalar multiples of each other.

Two lines **intersect** if there exist values λ and μ satisfying both equations simultaneously.

Two lines are **skew** if they are not parallel and do not intersect.

The **angle** θ between two lines with direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ satisfies $\cos\theta = \dfrac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|}$ (using the absolute value ensures $\theta \in [0°, 90°]$).

Question 1 Easy [4 marks]

Write down the vector equation of the line passing through $A(1, 2, -1)$ and $B(3, 0, 4)$. Find the coordinates of the point on the line where $\lambda = 3$.

MISTAKE ANALYSIS

Direction vector: $\overrightarrow{AB} = B - A = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$. *Line:* $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$. *At* $\lambda = 3$: $\begin{pmatrix} 1+6 \\ 2-6 \\ -1+15 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 14 \end{pmatrix}$, *i.e.* $(7, -4, 14)$. *Students who compute* $\overrightarrow{AB} = A - B = \begin{pmatrix} -2 \\ 2 \\ -5 \end{pmatrix}$ *get the opposite direction vector. Both* $\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$ *and* $\begin{pmatrix} -2 \\ 2 \\ -5 \end{pmatrix}$ *describe the same line – but the point at* $\lambda = 3$ *will differ. Always compute direction as* $B - A$ *(terminal minus initial), or verify the direction by checking that* A *and* B *both satisfy the equation.*

Question 2

Easy

[4 marks]

Determine whether the following two lines are parallel, intersecting, or skew.

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}.$$

MISTAKE ANALYSIS

Direction vectors: $\mathbf{d}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ *and* $\mathbf{d}_2 = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. *Since* $\mathbf{d}_2 = 2\mathbf{d}_1$, *the lines are parallel (or coincident). Check if* L_1 *and* L_2 *share a point: does* $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ *lie on* L_1*?* $1 + 2\lambda = 3$, $\lambda = 1$; $0 + 1 = 1$ ✓; $2 - 1 = 1 \neq 0$. *No. So the lines are parallel but not coincident (distinct parallel lines). Students who stop at "*$\mathbf{d}_2 = 2\mathbf{d}_1$*, therefore parallel" without checking coincidence lose the mark for the full description. Parallel lines may be coincident (the same line) – check by substituting a point from one line into the other.*

Question 3

Easy–Medium

[5 marks]

Find the angle between the lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

Give your answer in degrees, correct to 1 decimal place.

MISTAKE ANALYSIS

$\mathbf{d}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $|\mathbf{d}_1| = 3$. $\mathbf{d}_2 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, $|\mathbf{d}_2| = 3$. $\mathbf{d}_1 \cdot \mathbf{d}_2 = 2 - 2 + 4 = 4$. $\cos\theta = \dfrac{|4|}{3 \times 3} = \dfrac{4}{9}$. $\theta = \arccos\left(\dfrac{4}{9}\right) \approx 63.6°$. *Students who compute* $\cos\theta = \dfrac{\mathbf{d}_1 \cdot \mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}$ *without the absolute value and get a negative result* ($\cos\theta < 0$) *will find an obtuse angle. The angle between two lines is always taken as the acute angle* *(or right angle): use* $|\mathbf{d}_1 \cdot \mathbf{d}_2|$*, not* $\mathbf{d}_1 \cdot \mathbf{d}_2$.

Question 4 Medium [5 marks]

The line L has equation $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(a) Show that the point $P(4, 3, 1)$ lies on L.

(b) Find the point Q on L such that $\overrightarrow{OQ}$ is perpendicular to the direction of L.

MISTAKE ANALYSIS

(a) $\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$*:* $\lambda = 2$ *gives* $2+2=4$, $-1+4=3$, $3-2=1$. ✓ *(b) A general point on* L *is* $Q = \begin{pmatrix} 2+\lambda \\ -1+2\lambda \\ 3-\lambda \end{pmatrix}$. $\overrightarrow{OQ} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$*:* $(2+\lambda) + 2(-1+2\lambda) + (-(3-\lambda)) = 0$. $2 + \lambda - 2 + 4\lambda - 3 + \lambda = 0$, *so* $6\lambda = 3$, $\lambda = \frac{1}{2}$. $Q = \begin{pmatrix} 5/2 \\ 0 \\ 5/2 \end{pmatrix}$. *Students who set* $\overrightarrow{OQ} \cdot \overrightarrow{OP} = 0$ *(using the fixed point* P *instead of the direction vector) find a different condition and a wrong answer. The perpendicularity condition is between* $\overrightarrow{OQ}$ *(the position vector of the foot of the perpendicular from the origin to* L*) and the direction of* L *– not between* $\overrightarrow{OQ}$ *and* $\overrightarrow{OP}$*.*

Question 5 Medium [6 marks]

Determine whether the lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

intersect. If they do, find the point of intersection. If they do not, determine whether they are parallel or skew.

MISTAKE ANALYSIS

Direction vectors $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ *and* $\begin{pmatrix}1\\-1\\2\end{pmatrix}$*: not scalar multiples – not parallel. Set equations equal:* $1+2\lambda = 3+\mu$, $2+\lambda = 3-\mu$, $3-\lambda = 1+2\mu$. *From equations 1 and 2: add to eliminate* μ: $(1+2\lambda)+(2+\lambda) = (3+\mu)+(3-\mu)$, *so* $3+3\lambda = 6$, $\lambda = 1$. *Then* $\mu = 1+2-3 = -0\ldots$ *wait. Eq 1:* $1+2 = 3+\mu$, *so* $\mu = 0$. *Eq 2:* $2+1 = 3-0 = 3$ ✓. *Eq 3:* $3-1 = 2$ *and* $1+0 = 1$. $2 \neq 1$. *Contradiction. Lines do not intersect and are not parallel: they are* ***skew****. Students who check only two of the three equations and conclude the lines intersect are making the most common error in this type of problem. A consistent solution requires all three equations to be satisfied simultaneously. Always check the third equation after finding* λ *and* μ *from the first two.*

Question 6 Medium [7 marks]

The line L_1 passes through $A(1,0,2)$ with direction vector $\begin{pmatrix}2\\1\\-1\end{pmatrix}$. The line L_2 passes through $C(3,1,1)$ with direction vector $\begin{pmatrix}1\\-1\\2\end{pmatrix}$.

(a) Write down vector equations for L_1 and L_2.

(b) Show that L_1 and L_2 intersect and find the point of intersection.

(c) Find the acute angle between L_1 and L_2.

MISTAKE ANALYSIS

(a) L_1: $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$. L_2: $\mathbf{r} = \begin{pmatrix}3\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$. *(b) Set equal:* $1+2\lambda = 3+\mu$, $\lambda = 1+\mu\ldots$ *wait, Eq 2:* $\lambda = -1+\mu\ldots$ *Systematically: Eq 1:* $1+2\lambda = 3+\mu$ $(\mu = 2\lambda-2)$*; Eq 2:* $\lambda = 1+\mu = 1+2\lambda-2 = 2\lambda-1$, *so* $\lambda = 1$; $\mu = 0$. *Check Eq 3:* $2-1 = 1$ *and* $1+0 = 1$ ✓. *Intersection at* $\lambda = 1$: $(3,1,1)$. *(c)* $\mathbf{d}_1 \cdot \mathbf{d}_2 = 2-1-2 = -1$; $|\mathbf{d}_1| = |\mathbf{d}_2| = \sqrt{6}$. $\cos\theta = \frac{1}{6}$; $\theta \approx 80.4°$. *Students who fail to check the third equation after finding* λ *and* μ *from the first two may incorrectly claim intersection when the lines are skew. Always verify all three equations. In this question all three are consistent, confirming the intersection is genuine.*

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

Direction $\overrightarrow{AB} = B - A$	$\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$	**M1**
Line equation	$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$	**A1**
At $\lambda = 3$	$\begin{pmatrix} 1+6 \\ 2-6 \\ -1+15 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 14 \end{pmatrix}$	**A1**

Final answer: $(7, -4, 14)$

Solution – Question 2

$\mathbf{d}_2 = 2\mathbf{d}_1$: lines parallel	(or coincident – must check)	**M1**
Test $(3, 1, 0)$ on L_1: $\lambda = 1$ gives $(3, 1, 1) \neq (3, 1, 0)$	Not coincident	**R1**
Conclusion	Distinct parallel lines	**A1**

Final answer: Distinct parallel lines

Solution – Question 3

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 2 - 2 + 4 = 4$; $\lvert\mathbf{d}_1\rvert = \lvert\mathbf{d}_2\rvert = 3$	$\cos\theta = \dfrac{4}{9}$	**M1**
Take acute angle	$\theta = \arccos\left(\frac{4}{9}\right) \approx 63.6^\circ$	**A1**

Final answer: $\theta \approx 63.6^\circ$

Solution – Question 4

(a) At $\lambda = 2$: $\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$ ✓

(b)

$Q \begin{pmatrix} 2+\lambda \\ -1+2\lambda \\ 3-\lambda \end{pmatrix}$;	$=$	$(2+\lambda) + 2(-1+2\lambda) - (3-\lambda) = 0$	**M1**
$\overrightarrow{OQ} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 0$			
Simplify: $6\lambda - 3 = 0$		$\lambda = \frac{1}{2}$	**A1**
Q		$Q = \left(\frac{5}{2}, 0, \frac{5}{2}\right)$	**A1**

Final answer: $Q = \left(\frac{5}{2}, 0, \frac{5}{2}\right)$

Solution – Question 5

Not parallel (direction vectors not proportional)		**R1**
From equations 1 and 2: $\lambda = 1$, $\mu = 0$		**M1**
Check equation 3: $3 - 1 = 2$ vs $1 + 0 = 1$	$2 \neq 1$: no intersection	**R1**
Not parallel, not intersecting	Skew lines	**A1**

Final answer: Skew lines

Solution – Question 6

(a)

L_1: $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$

L_2: $\mathbf{r} = \begin{pmatrix}3\\1\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$ **A1**

(b)

Eq 1: $1 + 2\lambda = 3 + \mu$; Eq 2: $\lambda = 1 + \mu$... **M1**

At $\lambda = 1$: L_1 gives $(3, 1, 1) = C$, so $\mu = 0$: both give $(3, 1, 1)$ — Intersection at $(3, 1, 1)$ ✓ **A1**

(c)

$\mathbf{d}_1 \cdot \mathbf{d}_2 = (2)(1)+(1)(-1)+(-1)(2) = -1$ — $|\mathbf{d}_1| = \sqrt{6}$, $|\mathbf{d}_2| = \sqrt{6}$ **M1**

$\cos\theta = \frac{|-1|}{6} = \frac{1}{6}$ — $\theta = \arccos\left(\frac{1}{6}\right) \approx 80.4°$ **A1**

Final answer: (b) $(3, 1, 1)$ (c) $\approx 80.4°$
