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Dot Product & Planes

Recognition Training – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 3 – Geometry & Trigonometry
Level	Medium → Hard
Questions	6
Total marks	33
Instructions	Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

The **dot product** (scalar product) of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta$. Two vectors are **perpendicular** if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

The equation of a **plane** with normal vector $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ passing through point (x_0, y_0, z_0) is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, i.e. $ax + by + cz = d$ where $d = ax_0 + by_0 + cz_0$.

The **angle between a line and a plane**: if $\mathbf{d}$ is the direction of the line and $\mathbf{n}$ is the normal to the plane, then $\sin\phi = \dfrac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$ where ϕ is the angle between the line and the plane.

The **angle between two planes** equals the angle between their normal vectors (or its supplement – take the acute angle).

Question 1 — Medium — [4 marks]

Find the angle between $\mathbf{a} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$, giving your answer in degrees correct to 1 decimal place. Hence determine whether $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, acute, or obtuse.

> **MISTAKE ANALYSIS**
>
> $\mathbf{a} \cdot \mathbf{b} = 3 + 0 + 8 = 11$. $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$. $\cos\theta = \dfrac{11}{15}$. $\theta = \arccos\left(\dfrac{11}{15}\right) \approx 42.8°$. *The angle is acute since* $\mathbf{a} \cdot \mathbf{b} = 11 > 0$. *Students who compute* $|\mathbf{a}| = \sqrt{1+4+4} = \sqrt{9} = 3$ *correctly but then write* $|\mathbf{b}| = \sqrt{9+0+16} = \sqrt{25} = 5$ *as* $|\mathbf{b}| = 6$ *have a mental arithmetic error from* $\sqrt{25} = 5$ *being missed. Also: the sign of* $\mathbf{a} \cdot \mathbf{b}$ *tells you the type of angle immediately – positive means acute, zero means perpendicular, negative means obtuse – without computing* arccos.

Question 2

Medium

[5 marks]

Find the equation of the plane passing through the points $A(1,0,2)$, $B(3,1,0)$, and $C(0,2,1)$.

MISTAKE ANALYSIS

Two direction vectors in the plane: $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ *and* $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$. *Normal:* $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (1)(-1)-(-2)(2) \\ (-2)(-1)-(2)(-1) \\ (2)(2)-(1)(-1) \end{pmatrix} = \begin{pmatrix} -1+4 \\ 2+2 \\ 4+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$. *Plane through* $A(1,0,2)$*:* $3(x-1)+4(y-0)+5(z-2)=0$, *i.e.* $3x+4y+5z=13$. *Students who use only one direction vector (e.g.* $\overrightarrow{AB}$ *as the normal) produce a plane that passes through* A *and* B *but not necessarily* C. *The normal must be perpendicular to the plane – i.e. perpendicular to two independent vectors in the plane. Use the cross product of two such vectors.*

Question 3

Medium

[5 marks]

Find the point of intersection of the line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and the plane $2x+y-z=5$.

MISTAKE ANALYSIS

Substitute $x=1+\lambda$, $y=2-\lambda$, $z=2\lambda$ *into* $2x+y-z=5$*:* $2(1+\lambda)+(2-\lambda)-2\lambda=5$. $2+2\lambda+2-\lambda-2\lambda=5$. $4-\lambda=5$, *so* $\lambda=-1$. *Point:* $\begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$. *Verify:* $2(0)+3-(-2)=5$ ✓. *Students who substitute the line into the plane but omit the parametric form – writing* $2x+y-z=5$ *directly without expressing* x, y, z *in terms of* λ *– cannot solve for* λ. *The substitution step is mandatory:* $x=1+\lambda$, $y=2-\lambda$, $z=0+2\lambda$ *before substituting.*

Question 4 Medium–Hard [6 marks]

Find the acute angle between the planes $\Pi_1 : 2x - y + 2z = 3$ and $\Pi_2 : x + 2y - 2z = 1$.

MISTAKE ANALYSIS

Normal vectors: $\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ *and* $\mathbf{n}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$. $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 2 - 4 = -4$. $|\mathbf{n}_1| = 3$, $|\mathbf{n}_2| = 3$. $\cos\theta = \frac{|-4|}{9} = \frac{4}{9}$. $\theta = \arccos\left(\frac{4}{9}\right) \approx 63.6°$. *The angle between the planes is the angle between their normals (taking the acute value). Students who forget the absolute value and get* $\cos\theta = -4/9$, *giving* $\theta \approx 116.4°$, *have found the obtuse angle between the normals. The acute angle between the planes is always* $\leq 90°$ – *use* $|\mathbf{n}_1 \cdot \mathbf{n}_2|$ *in the numerator.*

Question 5 Hard [6 marks]

Find the angle between the line $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ and the plane $x + 2y + 2z = 6$.

MISTAKE ANALYSIS

Direction vector: $\mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. *Normal to plane:* $\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. $\mathbf{d} \cdot \mathbf{n} = 1 + 2 - 2 = 1$. $|\mathbf{d}| = \sqrt{3}$, $|\mathbf{n}| = 3$. *Angle between line and plane:* $\sin\phi = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|} = \frac{1}{3\sqrt{3}}$. $\phi = \arcsin\left(\frac{1}{3\sqrt{3}}\right) \approx 11.1°$. *The critical error: students use* $\cos\phi = \frac{|\mathbf{d} \cdot \mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$, *computing the angle between the line and the normal. This gives* $\phi \approx 78.9°$ – *the complement of the correct answer. The angle between a line and a plane is measured from the plane to the line, not from the normal to the line. Use* sin, *not* cos, *for line-plane angle.*

Question 6 Hard [7 marks]

(a) Find the equation of the plane Π passing through $P(2, 1, -1)$ and perpendicular to the line

$$L:\ \mathbf{r} = \begin{pmatrix}0\\3\\1\end{pmatrix} + t\begin{pmatrix}1\\-2\\3\end{pmatrix}.$$

(b) Find the distance from the origin to the plane Π.

(c) Find the point where L meets Π.

MISTAKE ANALYSIS

(a) The plane is perpendicular to L, so the normal to Π is the direction of L: $\mathbf{n} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$. Plane through $P(2,1,-1)$: $(x-2) - 2(y-1) + 3(z+1) = 0$, i.e. $x - 2y + 3z = -3$. (b) Distance from origin $(0,0,0)$ to $ax + by + cz = d$: dist $= \dfrac{|d|}{\sqrt{a^2+b^2+c^2}} = \dfrac{3}{\sqrt{1+4+9}} = \dfrac{3}{\sqrt{14}} = \dfrac{3\sqrt{14}}{14}$. (c) Substitute L into Π: $(0+t) - 2(3-2t) + 3(1+3t) = -3$. $t - 6 + 4t + 3 + 9t = -3$, so $14t = 0$, $t = 0$. Point: $\begin{pmatrix}0\\3\\1\end{pmatrix}$. Students who use the direction of L as a vector in the plane (rather than the normal to the plane) produce a plane parallel to L, not perpendicular to it. The direction vector of a line is the normal to any plane perpendicular to that line.

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

$\mathbf{a}\cdot\mathbf{b} = 3+0+8=11$; $\lvert\mathbf{a}\rvert = 3$, $\lvert\mathbf{b}\rvert = 5$	$\cos\theta = \frac{11}{15}$	**M1**
Angle	$\theta = \arccos\left(\frac{11}{15}\right) \approx 42.8^\circ$	**A1**
$\mathbf{a}\cdot\mathbf{b} > 0$	Acute angle	**R1**

Final answer: $\theta \approx 42.8^\circ$ (acute)

Solution – Question 2

$\overrightarrow{AB} = \begin{pmatrix}2\\1\\-2\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}-1\\2\\-1\end{pmatrix}$		**M1**
Normal $\mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC}$	$\begin{pmatrix}(1)(-1)-(-2)(2)\\(-2)(-1)-(2)(-1)\\(2)(2)-(1)(-1)\end{pmatrix} = \begin{pmatrix}3\\4\\5\end{pmatrix}$	**M1**
Plane through $A(1,0,2)$	$3(x-1)+4y+5(z-2)=0$	**A1**
Simplify	$3x+4y+5z=13$	**A1**

Final answer: $3x+4y+5z=13$

Solution – Question 3

Substitute: $2(1+\lambda)+(2-\lambda)-2\lambda=5$	$4-\lambda=5 \Rightarrow \lambda=-1$	**M1**
Point on line at $\lambda=-1$	$\begin{pmatrix}0\\3\\-2\end{pmatrix}$	**A1**
Verify: $0+3+2=5$	✓	**R1**

Final answer: $(0, 3, -2)$

Solution – Question 4

$\mathbf{n}_1 = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad |\mathbf{n}_1| = |\mathbf{n}_2| = 3$ **M1**

$\mathbf{n}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix};$

$\mathbf{n}_1 \cdot \mathbf{n}_2 = -4$

$\cos\theta = \frac{|-4|}{9} = \frac{4}{9} \quad \theta = \arccos\left(\frac{4}{9}\right) \approx 63.6^\circ$ **A1**

Final answer: $\theta \approx 63.6^\circ$

Solution – Question 5

$\mathbf{d} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \mathbf{n} = \quad |\mathbf{d}| = \sqrt{3}, \ |\mathbf{n}| = 3$ **M1**

$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}; \mathbf{d} \cdot \mathbf{n} = 1$

$\sin\phi = \frac{1}{3\sqrt{3}} \quad \phi = \arcsin\left(\frac{1}{3\sqrt{3}}\right) \approx 11.1^\circ$ **A1**

Final answer: $\phi \approx 11.1^\circ$

Solution – Question 6

(a) Normal direction of L: $\mathbf{n} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ = $(x-2) - 2(y-1) + 3(z+1) = 0$ **M1**

Simplify $x - 2y + 3z = -3$ **A1**

(b) Distance from origin $\frac{|-3|}{\sqrt{14}} = \frac{3\sqrt{14}}{14}$ **M1**

(c) Sub L into Π: $(0+t)-2(3-2t)+3(1+3t) = -3$ $\quad 14t = 0 \ \Rightarrow \ t = 0$ **M1**

Point at $t = 0$ $\quad \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ **A1**

Final answer: (a) $x - 2y + 3z = -3$ (b) $\dfrac{3\sqrt{14}}{14}$ (c) $(0, 3, 1)$