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Cross Product & Distances

Recognition Training – Set III

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 3 – Geometry & Trigonometry
Level	Medium → Hard
Questions	6
Total marks	35
Instructions	Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

The **cross product** (vector product) of $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$. Key properties: $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$; $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$; $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.

Area of triangle with sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$: Area $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$.

Distance from point P **to line** through A with direction $\mathbf{d}$: dist $= \dfrac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|}$.

Distance between skew lines: if $\mathbf{c} = \mathbf{b}_1 - \mathbf{b}_2$ (vector joining points on each line), dist $= \dfrac{|\mathbf{c} \cdot (\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|}$.

Question 1 — Medium — [5 marks]

Find $\mathbf{a} \times \mathbf{b}$ where $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$. Verify that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

MISTAKE ANALYSIS

$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (1)(2) - (-1)(3) \\ (-1)(1) - (2)(2) \\ (2)(3) - (1)(1) \end{pmatrix} = \begin{pmatrix} 2+3 \\ -1-4 \\ 6-1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$. *Verify:* $(5)(2) + (-5)(1) + (5)(-1) = 10 - 5 - 5 = 0$ ✓. $(5)(1) + (-5)(3) + (5)(2) = 5 - 15 + 10 = 0$ ✓. *Students who use the formula* $\begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$ *but assign the components incorrectly – especially the middle term sign – get the second component wrong. The middle component is* $a_3b_1 - a_1b_3$ *(note the reversed order:* a_3b_1*, not* a_1b_3*). The sign pattern is* $+, -, +$ *for the three components.*

Question 2 Medium [5 marks]

Find the area of the triangle with vertices $A(1,0,0)$, $B(0,2,0)$, and $C(0,0,3)$.

MISTAKE ANALYSIS

$\overrightarrow{AB} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}-1\\0\\3\end{pmatrix}$. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}(2)(3)-(0)(0)\\(0)(-1)-(-1)(3)\\(-1)(0)-(2)(-1)\end{pmatrix} = \begin{pmatrix}6\\3\\2\end{pmatrix}$. $|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{36+9+4} = 7$. *Area* $= \frac{1}{2} \times 7 = \frac{7}{2}$. *Students who compute the area as* $|\overrightarrow{AB} \times \overrightarrow{AC}| = 7$ *(without the factor of* $\frac{1}{2}$*) give the area of the parallelogram, not the triangle. The cross product magnitude gives the parallelogram area; the triangle is half of this.*

Question 3 Medium [5 marks]

Find the distance from the point $P(3,1,2)$ to the line $L: \mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda \begin{pmatrix}2\\1\\-1\end{pmatrix}$.

MISTAKE ANALYSIS

$\overrightarrow{AP} = P - A = \begin{pmatrix}2\\1\\1\end{pmatrix}$ *where* $A = (1,0,1)$. *Direction* $\mathbf{d} = \begin{pmatrix}2\\1\\-1\end{pmatrix}$, $|\mathbf{d}| = \sqrt{6}$. $\overrightarrow{AP} \times \mathbf{d} = \begin{pmatrix}(1)(-1)-(1)(1)\\(1)(2)-(2)(-1)\\(2)(1)-(1)(2)\end{pmatrix} = \begin{pmatrix}-2\\4\\0\end{pmatrix}$. $|\overrightarrow{AP} \times \mathbf{d}| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$. *Distance* $= \frac{2\sqrt{5}}{\sqrt{6}} = \frac{2\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{30}}{6} = \frac{\sqrt{30}}{3}$. *Students who compute the distance as* $|\overrightarrow{AP}| = \sqrt{6}$ *(the distance from* A *to* P *directly) have not projected onto the perpendicular direction. The formula* $\frac{|\overrightarrow{AP} \times \mathbf{d}|}{|\mathbf{d}|}$ *accounts for the component of* $\overrightarrow{AP}$ *perpendicular to the line.*

Question 4 Medium–Hard [6 marks]

(a) Show that the lines $L_1 : \mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix}$ are skew.

(b) Find the shortest distance between L_1 and L_2.

MISTAKE ANALYSIS

(a) Directions $\mathbf{d}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ *and* $\mathbf{d}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$*: not proportional, so not parallel. Set components equal: Eq 1:* $1 + \lambda = 1$, *so* $\lambda = 0$. *Eq 2:* $0 + 0 = 0 + \mu$, *so* $\mu = 0$. *Eq 3:* $0 = 1 + 0 = 1$. *Contradiction. Lines are not parallel and do not intersect: skew.* ✓ *(b)* $\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$. *Vector joining the two base points:* $\mathbf{c} = (1,0,1) - (1,0,0) = \begin{pmatrix}0\\0\\1\end{pmatrix}$. *Distance* $= \dfrac{|\mathbf{c}\cdot(\mathbf{d}_1 \times \mathbf{d}_2)|}{|\mathbf{d}_1 \times \mathbf{d}_2|} = \dfrac{|0+0+1|}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$. *The error: students who compute* $|\mathbf{c}|/|\mathbf{d}_1 \times \mathbf{d}_2|$ *(dividing the length of* $\mathbf{c}$ *by the length of the cross product) get a dimensionally inconsistent formula. The correct formula uses the scalar projection of* $\mathbf{c}$ *onto* $\mathbf{d}_1 \times \mathbf{d}_2$ *via the dot product – giving the component of* $\mathbf{c}$ *in the direction perpendicular to both lines.*

Question 5 Hard [7 marks]

The tetrahedron $ABCD$ has vertices $A(1,2,0)$, $B(3,0,1)$, $C(2,1,3)$, $D(4,4,2)$.

(a) Find the area of face ABC.

(b) Find the volume of the tetrahedron, given that the volume equals $\frac{1}{6}\left|(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}\right|$.

MISTAKE ANALYSIS

$\overrightarrow{AB} = \begin{pmatrix}2\\-2\\1\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}1\\-1\\3\end{pmatrix}$, $\overrightarrow{AD} = \begin{pmatrix}3\\2\\2\end{pmatrix}$. $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}(-2)(3)-(1)(-1)\\(1)(1)-(2)(3)\\(2)(-1)-(-2)(1)\end{pmatrix} = \begin{pmatrix}-5\\-5\\0\end{pmatrix}$. *Area of ABC* $= \frac{1}{2}\left|\begin{pmatrix}-5\\-5\\0\end{pmatrix}\right| = \frac{1}{2}\sqrt{50} = \frac{5\sqrt{2}}{2}$. *Scalar triple product:* $(-5)(3) + (-5)(2) + (0)(2) = -25$. *Volume* $= \frac{1}{6}\cdot 25 = \frac{25}{6}$. *Students who compute the area as* $|\overrightarrow{AB} \times \overrightarrow{AC}| = 5\sqrt{2}$ *(without halving) give the parallelogram area, not the triangle. Also: the scalar triple product* $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ *gives the signed volume of the parallelepiped; the tetrahedron volume is* $\frac{1}{6}$ *of the absolute value, not* $\frac{1}{3}$ *(which would give the pyramid volume if the base area were already computed).*

Question 6 Hard [7 marks]

The plane Π contains the line $L: \mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}1\\-1\\1\end{pmatrix}$ and the point $Q(0,2,1)$. Find:

(a) The equation of Π.

(b) The shortest distance from the point $R(3,0,2)$ to Π.

MISTAKE ANALYSIS

(a) The plane contains the direction $\mathbf{d} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ *(from line L) and also the vector* $\overrightarrow{AQ}$ *where* $A = (1,1,0)$ *is a point on L:* $\overrightarrow{AQ} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$. *Normal:* $\mathbf{n} = \mathbf{d} \times \overrightarrow{AQ} = \begin{pmatrix}1\\-1\\1\end{pmatrix} \times \begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}(-1)(1)-(1)(1)\\(1)(-1)-(1)(1)\\(1)(1)-(-1)(-1)\end{pmatrix} = \begin{pmatrix}-2\\-2\\0\end{pmatrix}$.

Simplify: $\mathbf{n} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ *(dividing by* -2*). Plane through* $A(1,1,0)$*:* $1(x-1)+1(y-1)+0=0$*, i.e.* $x+y=2$.

(b) Distance from $R(3,0,2)$ *to* $x+y=2$*: dist* $= \dfrac{|3+0-2|}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$. *The error: students who use* $\overrightarrow{AQ}$ *and the direction of L in the correct order but then forget to simplify the normal before computing the plane equation – using* $\begin{pmatrix}-2\\-2\\0\end{pmatrix}$ *directly – get the same plane* $-2x-2y=2(2)$*, i.e.* $x+y=2$ *after simplification. Any scalar multiple of the normal gives the same plane. However, the distance formula requires the normal used to match the coefficients in the plane equation – or simplify first.*

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (1)(2)-(-1)(3) \\ (-1)(1)-(2)(2) \\ (2)(3)-(1)(1) \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$ **M1**

Verify $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a}$ $(5)(2)+(-5)(1)+(5)(-1)=0$ ✓ **R1**

Verify $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}$ $(5)(1)+(-5)(3)+(5)(2)=0$ ✓ **R1**

Final answer: $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$

Solution – Question 2

$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$ $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ **M1**

$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{36+9+4} = 7$ $\text{Area} = \frac{1}{2} \times 7 = \frac{7}{2}$ **A1**

Final answer: $\text{Area} = \dfrac{7}{2}$

Solution – Question 3

$\overrightarrow{AP} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{d} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ $\overrightarrow{AP} \times \mathbf{d} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$ **M1**

$|\overrightarrow{AP} \times \mathbf{d}| = 2\sqrt{5}$, $|\mathbf{d}| = \sqrt{6}$ $\text{dist} = \dfrac{2\sqrt{5}}{\sqrt{6}} = \dfrac{\sqrt{30}}{3}$ **A1**

Final answer: $\dfrac{\sqrt{30}}{3}$

Solution – Question 4

(a)

Not parallel: $\mathbf{d}_1, \mathbf{d}_2$ not proportional		**R1**
Eq 1: $1+\lambda=1$, $\lambda=0$; Eq 2: $0=\mu$; Eq 3: $0=1+0=1$	$0 \neq 1:$ no intersection	**M1**
Not parallel, not intersecting	Skew ✓	**A1**

(b)

$\mathbf{d}_1 \times \mathbf{d}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$	$\lvert\mathbf{d}_1 \times \mathbf{d}_2\rvert = \sqrt{3}$	**M1**
$\mathbf{c} = \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$	$\mathbf{c}\cdot(\mathbf{d}_1 \times \mathbf{d}_2) = 0+0+1=1$	**M1**
Distance	$\dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$	**A1**

Final answer: (b) Distance $= \dfrac{\sqrt{3}}{3}$

Solution – Question 5

$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$	$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -5 \\ -5 \\ 0 \end{pmatrix}$	**M1**
Area of ABC	$\frac{1}{2}\sqrt{25+25+0} = \dfrac{5\sqrt{2}}{2}$	**A1**
$\overrightarrow{AD} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$; scalar triple product	$(-5)(3) + (-5)(2) + 0 = -25$	**M1**
Volume	$\frac{1}{6}\lvert -25 \rvert = \dfrac{25}{6}$	**A1**

Final answer: Area of $ABC = \dfrac{5\sqrt{2}}{2}$; Volume $= \dfrac{25}{6}$

Solution – Question 6

(a)

$\overrightarrow{AQ} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; normal $\mathbf{n} = \mathbf{d} \times \overrightarrow{AQ}$	$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$	**M1**
Plane through $A(1,1,0)$	$(x-1) + (y-1) = 0 \Rightarrow x + y = 2$	**A1**

(b)

Distance from $R(3,0,2)$ to $x+y=2$	$\dfrac{\lvert 3+0-2 \rvert}{\sqrt{1^2+1^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$	**M1**

Final answer: (a) $x + y = 2$ (b) $\dfrac{\sqrt{2}}{2}$
