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Trigonometry

Mistake Analysis – Set III

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 3 – Trigonometry
Level	Medium → Hard
Questions	6
Marks	36 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

$R\sin(x-\alpha)$: use $a\sin x - b\cos x = R\sin(x-\alpha)$ where $R = \sqrt{a^2+b^2}$, $\tan\alpha = b/a$.

Exact values: $\sin 30 = \frac{1}{2}$, $\cos 30 = \frac{\sqrt{3}}{2}$, $\sin 45 = \cos 45 = \frac{\sqrt{2}}{2}$, $\sin 60 = \frac{\sqrt{3}}{2}$, $\cos 60 = \frac{1}{2}$.

Identity proofs: work from one side to the other (do not work on both sides simultaneously). State clearly which side you are transforming.

Question 1

Medium [5 marks]

Express $5\sin x - 12\cos x$ in the form $R\sin(x-\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. State the minimum value of $5\sin x - 12\cos x$ and the value of $x \in [0, 2\pi]$ at which it occurs.

MISTAKE ANALYSIS

$R\sin(x-\alpha) = R\sin x\cos\alpha - R\cos x\sin\alpha$. Match: $R\cos\alpha = 5$ and $R\sin\alpha = 12$. $R = \sqrt{25+144} = 13$; $\tan\alpha = \frac{12}{5}$, so $\alpha = \arctan\frac{12}{5} \approx 1.176$ rad. $5\sin x - 12\cos x = 13\sin(x-\alpha)$. Minimum value: -13, when $\sin(x-\alpha) = -1$, i.e. $x - \alpha = \frac{3\pi}{2}$, so $x = \frac{3\pi}{2} + \alpha \approx 5.905$ rad. Students who write $R\sin(x+\alpha)$ for a difference $a\sin x - b\cos x$ use the wrong form. $R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$ requires both coefficients positive. A difference $a\sin x - b\cos x$ matches $R\sin(x-\alpha)$.

Question 2

Medium [5 marks]

Prove the identity $\dfrac{1}{1+\cos x} + \dfrac{1}{1-\cos x} \equiv \dfrac{2}{\sin^2 x}$.

MISTAKE ANALYSIS

LHS: combine over common denominator $(1+\cos x)(1-\cos x) = 1-\cos^2 x = \sin^2 x$. $\frac{(1-\cos x)+(1+\cos x)}{\sin^2 x} = \frac{2}{\sin^2 x}$ = *RHS.* ✓ *Students who multiply out the numerator incorrectly, writing* $(1+\cos x)+(1-\cos x) = 1$ *(forgetting the 2), produce* $\frac{1}{\sin^2 x}$ *rather than* $\frac{2}{\sin^2 x}$. *Also: the key identity is* $(1+\cos x)(1-\cos x) = 1-\cos^2 x = \sin^2 x$. *If this is not recognised, the proof stalls.*

Question 3 Medium [5 marks]

Solve $\sin\left(x+\dfrac{\pi}{6}\right) = \cos x$ for $0 \le x \le 2\pi$.

MISTAKE ANALYSIS

Expand: $\sin x\cos\frac{\pi}{6}+\cos x\sin\frac{\pi}{6} = \cos x$. $\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x = \cos x$. $\frac{\sqrt{3}}{2}\sin x = \frac{1}{2}\cos x$. $\tan x = \frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6}$ *or* $x = \pi+\frac{\pi}{6} = \frac{7\pi}{6}$. *Verify:* $\sin(\pi/6+\pi/6) = \sin(\pi/3) = \frac{\sqrt{3}}{2} = \cos(\pi/6)$ ✓. *Students who convert* $\cos x = \sin(\pi/2-x)$ *and then equate arguments miss the full solution set.* $\sin A = \sin B$ *gives* $A = B+2n\pi$ *or* $A = \pi-B+2n\pi$*: both families must be checked.*

Question 4 Medium–Hard [5 marks]

Prove the identity $\dfrac{\cos x-\sin x}{\cos x+\sin x} \equiv \dfrac{1-\tan x}{1+\tan x}$.

MISTAKE ANALYSIS

LHS: divide numerator and denominator by $\cos x$ *(valid where* $\cos x \neq 0$*):* $\dfrac{\cos x-\sin x}{\cos x+\sin x} = \dfrac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}} = \dfrac{1-\tan x}{1+\tan x}$ = *RHS.* ✓ *Students who multiply numerator and denominator by* $\frac{1}{\cos x}$ *(same operation, different notation) arrive at the same result. The key step is recognising that dividing by* $\cos x$ *converts* $\sin x/\cos x$ *into* $\tan x$. *Do not cross-multiply and attempt to prove* $(\cos x-\sin x)(1+\tan x) = (\cos x+\sin x)(1-\tan x)$*: this is circular (assuming what is to be proved).*

Question 5 Hard [8 marks]

Solve $3\cos^2 x = 2 - \sin x$ for $0 \le x \le 2\pi$. Give solutions in exact form where possible, otherwise to 3 significant figures.

MISTAKE ANALYSIS

Substitute $\cos^2 x = 1 - \sin^2 x$*:* $3(1-\sin^2 x) = 2 - \sin x$. $3 - 3\sin^2 x = 2 - \sin x \Rightarrow 3\sin^2 x - \sin x - 1 = 0$. *Quadratic in* $\sin x$*:* $\sin x = \dfrac{1 \pm \sqrt{13}}{6}$. $\sin x = \dfrac{1+\sqrt{13}}{6} \approx 0.7676$*:* $x \approx 0.874$ *or* $x \approx \pi - 0.874 \approx 2.268$. $\sin x = \dfrac{1-\sqrt{13}}{6} \approx -0.4343$*:* $x \approx \pi + 0.449 \approx 3.591$ *or* $x \approx 2\pi - 0.449 \approx 5.834$. *All four values lie in* $[0, 2\pi]$ ✓. *Students who use* $\cos^2 x = \cos^2 x$ *(not substituting via* $1 - \sin^2 x$*) cannot reduce to a single trig function. The substitution* $\cos^2 x = 1 - \sin^2 x$ *is essential to convert the equation to a quadratic in* $\sin x$.

Question 6 Hard [8 marks]

(a) Using the compound angle formula, find the exact value of $\sin\dfrac{\pi}{12}$.

(b) Using the result of (a), find the exact value of $\sin\dfrac{5\pi}{12}$.

(c) Hence evaluate $\sin\dfrac{\pi}{12} + \sin\dfrac{5\pi}{12}$ and simplify.

MISTAKE ANALYSIS

(a) $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$. $\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6} - \cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$. *(b)* $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$. $\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\sqrt{6}+\sqrt{2}}{4}$ *(same calculation with + sign). Alternatively:* $\sin\frac{5\pi}{12} = \cos\frac{\pi}{12}\ldots$ *but use compound angle directly. (c)* $\dfrac{\sqrt{6}-\sqrt{2}}{4} + \dfrac{\sqrt{6}+\sqrt{2}}{4} = \dfrac{2\sqrt{6}}{4} = \dfrac{\sqrt{6}}{2}$. *Students who use a calculator to evaluate* $\sin(\pi/12) \approx 0.259$ *and then try to reverse-engineer the exact form lose the A1 marks. The compound angle formula gives the exact result directly.*

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

$R\cos\alpha = 5$, $R\sin\alpha = 12$; $R = 13$, $\alpha = \arctan(12/5)$	$13\sin(x-\alpha)$	**M1**
Min $= -13$ at $x = 3\pi/2 + \alpha \approx 5.905$		**A1**

Solution – Question 2

Common denominator $(1+\cos x)(1-\cos x) = \sin^2 x$; numerator $= 2$	$\dfrac{2}{\sin^2 x}$ ✓	**M1**

Solution – Question 3

Expand: $\frac{\sqrt{3}}{2}\sin x = \frac{1}{2}\cos x$; $\tan x = \frac{1}{\sqrt{3}}$	$x = \dfrac{\pi}{6}$ or $\dfrac{7\pi}{6}$	**M1**

Solution – Question 4

Divide num and denom by $\cos x$	$\dfrac{1-\tan x}{1+\tan x}$ ✓	**M1**

Solution – Question 5

$\cos^2 x = 1 - \sin^2 x$: $3\sin^2 x - \sin x - 1 = 0$; $\sin x = \frac{1 \pm \sqrt{13}}{6}$ **M1**

Four solutions $x \approx 0.874,\ 2.268,\ 3.591,\ 5.834$ **A1**

Solution – Question 6

(a) $\pi/4 - \pi/6$: $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$ $\dfrac{\sqrt{6} - \sqrt{2}}{4}$ **M1**

(b) $\pi/4 + \pi/6$ $\dfrac{\sqrt{6} + \sqrt{2}}{4}$ **A1**

(c) Sum $\dfrac{\sqrt{6}}{2}$ **A1**

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