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Trigonometry

Mistake Analysis – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 3 – Trigonometry
Level	Medium → Hard
Questions	6
Marks	35 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

General solutions: $\sin\theta = k$: $\theta = \arcsin k + 2n\pi$ or $\pi - \arcsin k + 2n\pi$.
$\cos\theta = k$: $\theta = \pm\arccos k + 2n\pi$.
$\tan\theta = k$: $\theta = \arctan k + n\pi$.

Key identities: $\sin^2\theta + \cos^2\theta = 1$; $\tan\theta = \frac{\sin\theta}{\cos\theta}$; $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

Strategy for equations: factorise, substitute, or use identities to reduce to a single trig function. Never divide both sides by a trig function (you lose solutions); always factorise.

Question 1 — Medium — [5 marks]

Solve $2\cos^2 x - 3\cos x + 1 = 0$ for $0 \le x \le 2\pi$.

MISTAKE ANALYSIS

Let $u = \cos x$: $2u^2 - 3u + 1 = (2u-1)(u-1) = 0$. $\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$. $\cos x = 1 \Rightarrow x = 0$ or $x = 2\pi$. Solutions: $x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$. Students who use the quadratic formula (valid) often lose one solution by rounding $\arccos(1/2)$ and missing that $\cos\theta = k > 0$ has two solutions in $[0, 2\pi]$. $\cos x = \frac{1}{2}$ gives $x = \frac{\pi}{3}$ and $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Question 2 — Medium — [5 marks]

Solve $\sin 2x = \sin x$ for $0 \le x \le 2\pi$.

MISTAKE ANALYSIS

$2\sin x\cos x = \sin x \Rightarrow \sin x(2\cos x - 1) = 0$. $\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$. $\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$. *Solutions:* $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$. *The critical error: dividing both sides by* $\sin x$. *This loses the solutions* $\sin x = 0$ ($x = 0, \pi, 2\pi$)*. Never divide a trig equation by a trig expression. Factorise instead.* $\sin 2x = \sin x \Rightarrow \sin x(2\cos x - 1) = 0$*: both factors must be considered.*

Question 3 Medium [5 marks]

Prove that $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

MISTAKE ANALYSIS

$\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)} = \dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$. *Divide numerator and denominator by* $\cos A\cos B$*:* $= \dfrac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}} = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$. ✓*Students who try to derive this by adding* $\tan A + \tan B = \frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}$ *and then "simplifying" produce a rational expression that does not simplify to* $\tan(A+B)$ *by elementary means. The correct approach: start from* $\frac{\sin(A+B)}{\cos(A+B)}$ *and divide through by* $\cos A\cos B$.

Question 4 Medium–Hard [6 marks]

Prove the identity $\sin^2 x - \sin^2 y \equiv \sin(x+y)\sin(x-y)$.

MISTAKE ANALYSIS

Expand RHS: $\sin(x+y)\sin(x-y) = (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)$. $= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$. $= \sin^2 x(1-\sin^2 y) - (1-\sin^2 x)\sin^2 y$. $= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y = \sin^2 x - \sin^2 y$. ✓*Students who try to work from LHS find no obvious path unless they recognise* $\sin^2 x - \sin^2 y = (1-\cos^2 x) - (1-\cos^2 y) = \cos^2 y - \cos^2 x$*, which is also difficult to connect. Working from RHS and expanding is the cleanest approach.*

Question 5 Hard [6 marks]

Solve $\tan x = 2\sin x$ for $0 \le x \le 2\pi$.

MISTAKE ANALYSIS

$\frac{\sin x}{\cos x} = 2\sin x \Rightarrow \sin x\left(\frac{1}{\cos x} - 2\right) = 0$. $\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$. $\frac{1}{\cos x} = 2 \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$. *Solutions:* $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$. *Students who multiply both sides by* $\cos x$ *immediately write* $\sin x = 2\sin x \cos x$*, then* $\sin x = \sin 2x$*, which is the same equation reversed – but they now must solve* $\sin x - \sin 2x = 0$*:* $\sin x(1 - 2\cos x) = 0$*, giving* $x = 0, \pi, 2\pi$ *or* $\cos x = \frac{1}{2}$*. Note:* $x = 0$ *must be checked:* $\tan 0 = 0 = 2\sin 0$. ✓

Question 6 Hard [8 marks]

Given that $\tan\theta = -\dfrac{3}{4}$ and $\dfrac{\pi}{2} < \theta < \pi$:

(a) Find the exact values of $\sin\theta$ and $\cos\theta$.

(b) Find the exact values of $\sin 2\theta$ and $\cos 2\theta$.

(c) Hence find $\tan 2\theta$ and state the quadrant of 2θ.

MISTAKE ANALYSIS

(a) $\tan\theta = -\frac{3}{4}$*: in the second quadrant,* $\sin\theta > 0$ *and* $\cos\theta < 0$*. From* $\tan\theta = -3/4$*: opposite* $= 3$*, adjacent* $= 4$ *(ignoring signs), hypotenuse* $= 5$. $\sin\theta = \frac{3}{5}$*;* $\cos\theta = -\frac{4}{5}$*. (b)* $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{3}{5}\cdot\left(-\frac{4}{5}\right) = -\frac{24}{25}$. $\cos 2\theta = 1 - 2\sin^2\theta = 1 - \frac{18}{25} = \frac{7}{25}$*. (c)* $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-24/25}{7/25} = -\frac{24}{7}$*. Since* $\sin 2\theta < 0$ *and* $\cos 2\theta > 0$*:* 2θ *is in the fourth quadrant. Students who use* $\sin\theta = -3/5$ *(taking the wrong sign in Q2) get* $\sin 2\theta = +24/25$*, which places* 2θ *in a different quadrant. The quadrant of the original angle* θ *determines the signs of* $\sin\theta$ *and* $\cos\theta$ *– state these before computing double angle values.*

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

$(2\cos x - 1)(\cos x - 1) = 0$	$x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, 2\pi$	**M1**

Solution – Question 2

$\sin x(2\cos x - 1) = 0$	$x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$	**M1**
R1: never divide by $\sin x$ – factorisation preserves all solutions		**R1**

Solution – Question 3

$\frac{\sin(A+B)}{\cos(A+B)}$; divide numerator and denominator by $\cos A \cos B$	$\frac{\tan A + \tan B}{1 - \tan A \tan B}$ ✓	**M1**

Solution – Question 4

RHS: $(\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y) = \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$		**M1**
$= \sin^2 x(1 - \sin^2 y) - (1 - \sin^2 x)\sin^2 y = \sin^2 x - \sin^2 y$	✓	**A1**

Solution – Question 5

$\sin x(1/\cos x - 2) = 0$; $\sin x = 0$ or $\cos x = 1/2$	$x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$	**M1**

Solution – Question 6

(a) Q2: $\sin\theta = 3/5$, $\cos\theta = -4/5$		**M1**
(b) $\sin 2\theta = -24/25$; $\cos 2\theta = 7/25$		**A1**
(c) $\tan 2\theta = -24/7$; Q4 ($\sin < 0$, $\cos > 0$)		**A1**

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