Trigonometry

Mistake Analysis – Set I

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 3 – Trigonometry
Level	Easy → Medium
Questions	6
Marks	33 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Compound angle formulae: $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$; $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$.

Double angle formulae: $\sin 2A = 2 \sin A \cos A$; $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.

R-formula: $a \sin\theta + b\cos\theta = R\sin(\theta + \alpha)$ where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = b/a$.

General solutions: $\sin\theta = k \Rightarrow \theta = \arcsin k + 2n\pi$ or $\theta = \pi - \arcsin k + 2n\pi$.

Question 1

Easy [5 marks]

Given that $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{5}{13}$, where A and B are both acute angles, find the exact value of $\sin(A + B)$.

MISTAKE ANALYSIS

$\cos A = \frac{4}{5}$ *(since A is acute and $\sin^2 A + \cos^2 A = 1$).* $\sin B = \frac{12}{13}$ *(since B is acute).* $\sin(A+B) = \sin A \cos B + \cos A \sin B = \frac{3}{5} \cdot \frac{5}{13} + \frac{4}{5} \cdot \frac{12}{13} = \frac{15}{65} + \frac{48}{65} = \frac{63}{65}$. *The most common error: using the wrong sign.* $\sin(A+B) = \sin A \cos B + \cos A \sin B$ *(positive).* $\sin(A - B)$ *uses a minus sign. Memorise: "sine sum: same sign." Also: students who use* $\cos A = 3/5$ *(confusing* $\sin A$ *and* $\cos A$*) produce a different wrong answer. Read which ratio is given for which angle carefully.*

Question 2 Easy [4 marks]

(a) Expand $\cos(x - 30)$ in the form $p\cos x + q\sin x$.

(b) Hence find the exact value of $\cos 15$.

MISTAKE ANALYSIS

(a) $\cos(x-30) = \cos x\cos 30 + \sin x \sin 30 = \dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x$. *(b) Set* $x = 45$*:* $\cos 15 = \cos(45-30) = \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{2}}{2} + \dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{2} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$. *Students who use* $\cos(A-B) = \cos A\cos B - \sin A \sin B$ *(with a minus sign) get* $\cos(x+30)$ *instead.* $\cos(A-B)$ *has a plus sign:* $\cos A\cos B + \sin A\sin B$*. Memory aid: cosine is "opposite" – where sine has* $+$*, cosine has* $-$*, and vice versa.*

Question 3 Medium [5 marks]

(a) Write $\cos 2x$ in terms of $\cos x$ only.

(b) Given that $\cos x = \dfrac{3}{5}$ and $0 \le x \le \dfrac{\pi}{2}$, find the exact values of $\cos 2x$ and $\sin 2x$.

MISTAKE ANALYSIS

(a) $\cos 2x = 2\cos^2 x - 1$*. (b)* $\cos 2x = 2\left(\frac{3}{5}\right)^2 - 1 = \frac{18}{25} - 1 = -\frac{7}{25}$. $\sin x = \frac{4}{5}$ *(since* x *is in the first quadrant).* $\sin 2x = 2\sin x\cos x = 2\cdot\frac{4}{5}\cdot\frac{3}{5} = \frac{24}{25}$*. Students who use* $\cos 2x = \cos^2 x - \sin^2 x$ *and then compute* $\cos^2 x + \sin^2 x$ *(adding instead of subtracting) produce* $\cos 2x = 1$*. The formula requires subtraction. Also: three forms of* $\cos 2x$ *exist – choose the form that matches what you are given. Given* $\cos x$*: use* $\cos 2x = 2\cos^2 x - 1$*. Given* $\sin x$*: use* $\cos 2x = 1 - 2\sin^2 x$*.*

Question 4 Medium [6 marks]

Express $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$. State the maximum value of $3\sin x + 4\cos x$ and the value of x at which it occurs (for $0 \le x \le 2\pi$).

MISTAKE ANALYSIS

Expand $R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x \sin\alpha$*. Match:* $R\cos\alpha = 3$ *and* $R\sin\alpha = 4$*.* $R = \sqrt{9+16} = 5$*;* $\tan\alpha = \frac{4}{3}$*, so* $\alpha = \arctan\left(\frac{4}{3}\right) \approx 0.927$ *rad.* $3\sin x + 4\cos x = 5\sin(x+\alpha)$*. Maximum value: 5, when*

$\sin(x+\alpha) = 1$, *i.e.* $x + \alpha = \frac{\pi}{2}$, *so* $x = \frac{\pi}{2} - \alpha \approx 0.644$ *rad. The error: students who write* $\tan\alpha = \frac{3}{4}$ *(taking the ratio the wrong way) get* $\alpha = \arctan(3/4)$. *The matching equations are* $R\cos\alpha = a$ *(coefficient of* $\sin x$*) and* $R\sin\alpha = b$ *(coefficient of* $\cos x$*).* $\tan\alpha = \frac{R\sin\alpha}{R\cos\alpha} = \frac{b}{a} = \frac{4}{3}$.

Question 5 Medium [6 marks]

Solve each equation for $0 \le x \le 2\pi$:

(a) $2\sin^2 x - \sin x - 1 = 0$

(b) $\cos 2x = \cos x$

MISTAKE ANALYSIS

(a) Factor: $(2\sin x + 1)(\sin x - 1) = 0$. $\sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}, \frac{11\pi}{6}$. $\sin x = 1 \Rightarrow x = \frac{\pi}{2}$. *Solutions:* $x = \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$. *(b) Use* $\cos 2x = 2\cos^2 x - 1$*:* $2\cos^2 x - 1 = \cos x \Rightarrow 2\cos^2 x - \cos x - 1 = 0 \Rightarrow (2\cos x + 1)(\cos x - 1) = 0$. $\cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$*;* $\cos x = 1 \Rightarrow x = 0, 2\pi$. *Solutions:* $x = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, 2\pi$. *For (b): students who solve* $\cos 2x = \cos x$ *by writing* $2x = x$ *miss the full solution.* $\cos\theta = \cos\phi$ *means* $\theta = \pm\phi + 2n\pi$, *not* $\theta = \phi$. *The substitution approach (converting* $\cos 2x$ *to a quadratic in* $\cos x$*) is safer.*

Question 6 Medium [7 marks]

(a) Prove the identity $(\sin x + \cos x)^2 \equiv 1 + \sin 2x$.

(b) Hence or otherwise, prove that $\sin x + \cos x \equiv \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$.

(c) Find the maximum value of $\sin x + \cos x$ and the smallest positive x at which it occurs.

MISTAKE ANALYSIS

(a) $(\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + \sin 2x$. ✓ *(b) From (a):* $(\sin x + \cos x)^2 = 1 + \sin 2x = 1 + 2\sin x\cos x$. *Also:* $\sqrt{2}\sin(x + \pi/4) = \sqrt{2}(\sin x\cos(\pi/4) + \cos x\sin(\pi/4)) = \sqrt{2}\cdot\frac{\sqrt{2}}{2}(\sin x + \cos x) = \sin x + \cos x$. ✓ *(c) Maximum of* $\sqrt{2}\sin(x + \pi/4)$ *is* $\sqrt{2}$, *at* $x + \pi/4 = \pi/2$, *i.e.* $x = \pi/4$. *Students who attempt (b) by squaring both sides must verify that both sides are non-negative at the relevant* x *values – squaring can introduce extraneous solutions. The direct expansion of* $\sqrt{2}\sin(x + \pi/4)$ *using the compound angle formula is cleaner.*

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

$\cos A = 4/5,\ \sin B = 12/13;\ \sin(A+B) = \frac{3}{5}\cdot\frac{5}{13} + \frac{4}{5}\cdot\frac{12}{13} = \frac{15+48}{65}$ $\quad \dfrac{63}{65}$ **M1**

Solution – Question 2

(a) $\cos(x-30) = \cos x \cos 30 + \sin x \sin 30$ $\quad \dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x$ **M1**

(b) $x = 45$: $\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$ $\quad \dfrac{\sqrt{6}+\sqrt{2}}{4}$ **A1**

Solution – Question 3

(a) $\cos 2x = 2\cos^2 x - 1$ **M1**

(b) $\cos 2x = \frac{18}{25} - 1 = -\frac{7}{25};\ \sin x = \frac{4}{5};\ \sin 2x = \frac{24}{25}$ **A1**

Solution – Question 4

$R\cos\alpha = 3,\ R\sin\alpha = 4;\ R = 5,\ \alpha = \arctan(4/3)$ $\quad 5\sin(x + \arctan(4/3))$ **M1**

Max $= 5$ at $x = \frac{\pi}{2} - \arctan\frac{4}{3}$ **A1**

Solution – Question 5

(a) $(2\sin x + 1)(\sin x - 1) = 0$ $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$ **M1**

(b) $2\cos^2 x - \cos x - 1 = (2\cos x + 1)(\cos x - 1) = 0$ $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$ **M1**

Solution – Question 6

(a) $\sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + \sin 2x$ ✓ **M1**

(b) $\sqrt{2}\sin(x + \pi/4) = \sqrt{2} \cdot \frac{\sqrt{2}}{2}(\sin x + \cos x) = \sin x + \cos x$ ✓ **M1**

(c) Max $= \sqrt{2}$ at $x = \pi/4$ **A1**

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