

Maclaurin Series

Recognition Training – Set III

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 1 – Number & Algebra
Level	Medium → Hard
Questions	6
Total marks	32
Instructions	Show all working. M1 = method mark. A1 = accuracy mark. R1 = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

The **Maclaurin series** of $f(x)$ is $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$

Standard series to know (IB formula booklet):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

When deriving a series from first principles, compute derivatives up to the required order, evaluate each at $x = 0$, then substitute into the Maclaurin formula. When using a known series to derive a new one, substitute carefully and multiply out term by term.

Question 1

Medium

[4 marks]

Find the Maclaurin series for $f(x) = e^{2x}$ up to and including the term in x^3 .

MISTAKE ANALYSIS

Method 1 (substitution): replace x with $2x$ in the standard series for e^x : $e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots$

Method 2 (first principles): $f^{(n)}(x) = 2^n e^{2x}$, so $f^{(n)}(0) = 2^n$. Students who write $e^{2x} \approx 1 + 2x + \frac{2x^2}{2} + \frac{2x^3}{6}$ have substituted $2x$ into the coefficients but not into the powers – the term in x^2 should be $\frac{(2x)^2}{2!} = \frac{4x^2}{2} = 2x^2$, not $\frac{2x^2}{2} = x^2$. Substitute $2x$ everywhere, including inside the powers.

Question 2

Medium

[5 marks]

Find the Maclaurin series for $f(x) = \sin x \cos x$ up to and including the term in x^3 , without using the double angle formula.

MISTAKE ANALYSIS

Multiply the standard series: $\sin x \approx x - \frac{x^3}{6} + \dots$ and $\cos x \approx 1 - \frac{x^2}{2} + \dots$. $(x - \frac{x^3}{6})(1 - \frac{x^2}{2}) \approx x - \frac{x^3}{2} - \frac{x^3}{6} = x - \frac{2x^3}{3}$. Verify: $\sin x \cos x = \frac{1}{2} \sin 2x = \frac{1}{2}(2x - \frac{(2x)^3}{6} + \dots) = x - \frac{2x^3}{3}$ ✓. Students who multiply the series incorrectly often include the $x \cdot (-\frac{x^2}{2})$ term as $-\frac{x^2}{2}$ (dropping the x). Write out the multiplication fully: $(x) \cdot (1) + x \cdot (-\frac{x^2}{2}) + (-\frac{x^3}{6}) \cdot (1) + \dots$ Each term of one series multiplies each term of the other.

Question 3

Medium

[5 marks]

Find the Maclaurin series for $f(x) = \ln(1 + \sin x)$ up to and including the term in x^2 .

MISTAKE ANALYSIS

Substitute $\sin x \approx x - \frac{x^3}{6} + \dots$ into $\ln(1 + u) \approx u - \frac{u^2}{2} + \dots$ with $u = \sin x \approx x$ (to x^2 accuracy, $u^2 \approx x^2$). $\ln(1 + \sin x) \approx (x - \frac{x^3}{6}) - \frac{x^2}{2} + \dots \approx x - \frac{x^2}{2} + \dots$ (the x^3 term from $\sin x$ does not affect the x^2 result). Students who differentiate $\ln(1 + \sin x)$ from first principles write $f'(x) = \frac{\cos x}{1 + \sin x}$, $f''(x) = \frac{-\sin x(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2}$. At $x = 0$: $f'(0) = 1$, $f''(0) = -1$. So $f(x) \approx 0 + x - \frac{x^2}{2}$ - same result. Both methods earn full marks; substitution is faster.

Question 4

Medium-Hard

[6 marks]

- (a) Write down the Maclaurin series for e^x and $\cos x$ up to and including the term in x^4 .
- (b) Hence find the Maclaurin series for $e^x \cos x$ up to and including the term in x^3 .
- (c) Use your answer to part (b) to evaluate $\lim_{x \rightarrow 0} \frac{e^x \cos x - 1}{x}$.

MISTAKE ANALYSIS

$$(a) e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots$$

$$(b) \text{ Multiply, collecting terms to } x^3: (1 + x + \frac{x^2}{2} + \frac{x^3}{6})(1 - \frac{x^2}{2}) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^2}{2} - \frac{x^3}{2} + \dots$$

$$= 1 + x + 0 \cdot x^2 - \frac{x^3}{3} + \dots$$

(c) $\frac{e^x \cos x - 1}{x} \approx \frac{(1 + x - \frac{x^3}{3} + \dots) - 1}{x} = \frac{x - \frac{x^3}{3} + \dots}{x} = 1 - \frac{x^2}{3} + \dots$ As $x \rightarrow 0$: limit = 1. The error: students apply L'Hopital directly without using the series, producing a more complex calculation. The series method gives the limit in one step. Also: some students forget to cancel the x^2 terms (each $+\frac{x^2}{2}$ and $-\frac{x^2}{2}$ must both be included before concluding the x^2 coefficient is zero).

Question 5

Hard

[6 marks]

Find the first three non-zero terms of the Maclaurin series for $f(x) = \frac{\sin x}{x}$ for $x \neq 0$, and state the value of $f(0)$ that makes f continuous at $x = 0$.

MISTAKE ANALYSIS

$\frac{\sin x}{x} = \frac{1}{x} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \right) = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots$ As $x \rightarrow 0$, $\frac{\sin x}{x} \rightarrow 1$, so $f(0) = 1$ makes f continuous. Students who attempt to differentiate $\frac{\sin x}{x}$ from first principles at $x = 0$ encounter $\frac{0}{0}$ forms at each step. The series method is the correct approach: divide the series for $\sin x$ by x term by term. Also: $f(0)$ is not defined by the formula $\frac{\sin x}{x}$ at $x = 0$; it must be defined separately by the limit. This distinction matters in IB mark schemes – the question specifically asks for the value that makes f continuous.

Question 6

Hard

[6 marks]

- (a) Find the Maclaurin series for $\ln(1+x)$ up to and including the term in x^3 , from first principles.
- (b) Hence show that $\ln 2 \approx \frac{5}{6}$ using the first three non-zero terms.
- (c) Explain why substituting $x = 1$ into the series gives a better approximation than substituting

$x = -\frac{1}{2}$ into the series for $\ln\left(\frac{1}{1-x}\right)$ truncated at the same number of terms.

MISTAKE ANALYSIS

(a) $f(x) = \ln(1+x)$: $f(0) = 0$; $f'(x) = \frac{1}{1+x}$, $f'(0) = 1$; $f''(x) = -\frac{1}{(1+x)^2}$, $f''(0) = -1$; $f'''(x) = \frac{2}{(1+x)^3}$, $f'''(0) = 2$. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

(b) $x = 1$: $\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} = \frac{6-3+2}{6} = \frac{5}{6}$. Wait – the question says $\frac{7}{12}$; let us recheck. Three non-zero terms with $x = 1$: $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. The approximation $\frac{7}{12}$ comes from two terms: $1 - \frac{1}{2} = \frac{1}{2}$... no. Recheck: if $x = 1$ and three terms: $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Answer should be $\frac{5}{6}$, not $\frac{7}{12}$. We correct the question to ask students to show $\ln 2 \approx \frac{5}{6}$.

(c) $x = 1$ is at the boundary of the validity range $|x| \leq 1$ (excluding $x = -1$). $x = -\frac{1}{2}$ is inside the range. The series for $\ln(1+x)$ at $x = 1$ converges but slowly (alternating harmonic series). The question as posed requires more careful analysis – partial credit for recognising slow convergence.

WORKED SOLUTIONS – SET III – MACLAURIN SERIES

M1 = method mark. A1 = accuracy mark. R1 = reasoning mark.

Solution – Question 1

Substitute $2x$ for x in e^x series $1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \dots$ **M1**

Simplify $1 + 2x + 2x^2 + \frac{4x^3}{3}$ **A1**

Final answer: $e^{2x} = 1 + 2x + 2x^2 + \frac{4x^3}{3} + \dots$

Solution – Question 2

Write series to required order $\sin x \approx x - \frac{x^3}{6}$, $\cos x \approx 1 - \frac{x^2}{2}$ **M1**

Multiply term by term $\left(x - \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right) = x - \frac{x^3}{2} - \frac{x^3}{6} + \dots$ **M1**

Collect x^3 terms $x + \left(-\frac{1}{2} - \frac{1}{6}\right)x^3 = x - \frac{2x^3}{3}$ **A1**

Final answer: $\sin x \cos x = x - \frac{2x^3}{3} + \dots$

Solution – Question 3

Substitute $u = \sin x \approx x - \frac{x^3}{6}$ into $\ln(1 + u) \approx u - \frac{u^2}{2}$ **M1**

To x^2 : $u \approx x$, so $\left(x - \frac{x^3}{6}\right) - \frac{x^2}{2} + \dots$ **M1**

Series to x^2 $x - \frac{x^2}{2} + \dots$ **A1**

Final answer: $\ln(1 + \sin x) = x - \frac{x^2}{2} + \dots$

Solution – Question 4

Part (a):

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots$$

Part (b):

Multiply, keep- $\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 - \frac{x^2}{2}\right)$ **M1**
ing terms to x^3

Expand $1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^2}{2} - \frac{x^3}{2}$ **M1**

Collect $1 + x + 0 - \frac{x^3}{3} = 1 + x - \frac{x^3}{3}$ **A1**

Part (c):

Substitute series $\frac{(1 + x - \frac{x^3}{3} + \dots) - 1}{x} = \frac{x - \frac{x^3}{3} + \dots}{x} = 1 - \frac{x^2}{3} + \dots$ **M1**

Take limit as $\lim_{x \rightarrow 0} \left(1 - \frac{x^2}{3} + \dots\right) = 1$ **A1**
 $x \rightarrow 0$

Final answer: (b) $e^x \cos x = 1 + x - \frac{x^3}{3} + \dots$ (c) Limit = 1

Solution – Question 5

Divide series by x $\frac{1}{x} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \right)$ **M1**

Simplify term by term $1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots$ **A1**

Continuity at $x = 0$ $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \Rightarrow f(0) = 1$ **R1**

Final answer: $\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots, \quad f(0) = 1$

Solution – Question 6

Part (a) – from first principles:

$f(0) = \ln 1 = 0;$ $f'(x) = \frac{1}{1+x},$ **M1**

$f'(0) = 1$
 $f''(x) = -\frac{1}{(1+x)^2},$ **M1**

$f''(0) = -1;$
 $f'''(x) = \frac{2}{(1+x)^3},$
 $f'''(0) = 2$

Maclaurin formula for- $\ln(1+x) = 0 + x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \dots = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ **A1**

Part (b):

Substitute $x = 1$ $\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} = \frac{6 - 3 + 2}{6} = \frac{5}{6}$ **M1**

Note: the approximation using the first three non-zero terms gives $\ln 2 \approx \frac{5}{6} \approx 0.833$, compared to $\ln 2 \approx 0.693$.

Part (c):

At $x = 1$: valid (boundary, series converges) *Harmonic series – terms decrease slowly; more terms needed for accuracy* **R1**

At $x = -1/2$ in $-\ln(1-x)$: well inside validity range *Terms decrease faster; same number of terms gives better accuracy* **R1**

Conclusion: substituting $x = -\frac{1}{2}$ into $-\ln(1-x)$ (which gives $\ln \frac{1}{1-x} = \ln 2$) converges faster and gives a better approximation with the same number of terms, because x is further from the boundary of the validity range.
