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Convergence, Binomial Theorem & Series

Recognition Training – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 1 – Number & Algebra
Level	Medium $\to$ Hard
Questions	6
Total marks	32
Instructions	Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

An infinite geometric series $\sum_{n=1}^{\infty} u_1 r^{n-1}$ converges if and only if $|r| < 1$, and its sum is $S_\infty = \frac{u_1}{1-r}$.

The **binomial theorem** for positive integer n: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

For fractional or negative index n, the expansion $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$ is valid for $|x| < 1$.

Question 1 Medium [4 marks]

A geometric series has first term 8 and common ratio r. Given that the series converges and $S_\infty = 24$, find r.

MISTAKE ANALYSIS

$S_\infty = \frac{u_1}{1-r} = 24$, so $\frac{8}{1-r} = 24$, giving $1 - r = \frac{1}{3}$, so $r = \frac{2}{3}$. Check: $|r| = \frac{2}{3} < 1$ ✓ – series converges.

The error: students solve correctly for r but forget to verify $|r| < 1$. If $r = \frac{2}{3}$ had satisfied the equation but $|r| \geq 1$, the series would not converge and the formula $S_\infty = \frac{u_1}{1-r}$ would be invalid. Always state the convergence check explicitly – it is a required step in IB mark schemes.

Question 2 Medium [5 marks]

Find the range of values of x for which the following series converges, and find its sum in terms of x.

$$\sum_{n=1}^{\infty}(2x-1)^n$$

MISTAKE ANALYSIS

This is a geometric series with first term $u_1 = (2x-1)$ and common ratio $r = (2x-1)$. Converges when $|2x-1| < 1$, i.e. $-1 < 2x-1 < 1$, so $0 < x < 1$. $S_\infty = \dfrac{2x-1}{1-(2x-1)} = \dfrac{2x-1}{2-2x} = \dfrac{2x-1}{2(1-x)}$. Students write $u_1 = 1$ (misreading the first term as the constant coefficient of the series). When $n = 1$: $u_1 = (2x-1)^1 = 2x-1$, not 1. Also: the convergence condition must be expressed as a range of x, not left as $|2x-1| < 1$ without solving for x.

Question 3 Medium [5 marks]

Find the coefficient of x^3 in the expansion of $(2+3x)^6$.

MISTAKE ANALYSIS

The term containing x^3 is $\binom{6}{3}(2)^3(3x)^3 = 20 \times 8 \times 27x^3 = 4320x^3$. Students make two errors here: (1) writing $\binom{6}{3}(2)^3(3)^3$ but then computing $\binom{6}{3} = \dfrac{6!}{3!} = 120$ instead of $\dfrac{6!}{3!\,3!} = 20$; (2) omitting the coefficient 3 from $(3x)^3$, writing $(x)^3$ instead of $(3x)^3$. The full term is $\binom{6}{k}a^{6-k}b^k$ where $a = 2$, $b = 3x$, $k = 3$. Both a and b must be substituted in full.

Question 4 Medium–Hard [6 marks]

(a) Write down the first three terms of the expansion of $\left(1+\dfrac{x}{2}\right)^8$ in ascending powers of x.

(b) Hence find an approximation to $(1.005)^8$, giving your answer correct to 5 decimal places.

MISTAKE ANALYSIS

(a) $\left(1+\frac{x}{2}\right)^8 = 1 + 8\cdot\frac{x}{2} + \binom{8}{2}\left(\frac{x}{2}\right)^2 + \cdots = 1 + 4x + 7x^2 + \cdots$

(b) Set $\frac{x}{2} = 0.005$, *so* $x = 0.01$. $(1.005)^8 \approx 1 + 4(0.01) + 7(0.01)^2 = 1 + 0.04 + 0.0007 = 1.04070$.

Students who set $x = 0.005$ *instead of* $x = 0.01$ *compute* $1+4(0.005)+7(0.005)^2 = 1.02018$ *– wrong because the expansion has* $x/2$*, not* x*, in the bracket. Read the bracket carefully:* $(1+x/2)^8$ *at* $x/2 = 0.005$ *means* $x = 0.01$.

Question 5 Hard [6 marks]

Find the first three terms of the expansion of $\frac{1}{\sqrt{1+2x}}$ in ascending powers of x, stating the range of values of x for which the expansion is valid.

MISTAKE ANALYSIS

Write as $(1+2x)^{-1/2}$. *Use* $(1+u)^n \approx 1 + nu + \frac{n(n-1)}{2!}u^2 + \cdots$ *with* $n = -\frac{1}{2}$, $u = 2x$. *Term 1:* 1. *Term 2:* $-\frac{1}{2}\cdot 2x = -x$. *Term 3:* $\frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(2x)^2 = \frac{3}{8}\cdot 4x^2 = \frac{3x^2}{2}$. *So* $(1+2x)^{-1/2} \approx 1 - x + \frac{3x^2}{2}$. *Valid for* $|2x| < 1$, *i.e.* $|x| < \frac{1}{2}$. *Students apply the expansion to* $(1+2x)^{-1/2}$ *treating* $u = x$ *instead of* $u = 2x$, *writing the second term as* $-\frac{1}{2}x$. *The substitution must be* $u = 2x$ *in full – every power of* u *carries the coefficient* 2.

Question 6 Hard [6 marks]

(a) Expand $(1+x)^{1/2}$ up to and including the term in x^2, stating the range of validity.

(b) By substituting a suitable value of x, find an approximation to $\sqrt{3}$, giving your answer as a fraction.

MISTAKE ANALYSIS

(a) $(1+x)^{1/2} \approx 1 + \frac{1}{2}x + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}x^2 = 1 + \frac{x}{2} - \frac{x^2}{8}$. *Valid for* $|x| < 1$.

(b) We want $\sqrt{3}$. *Write* $\sqrt{3} = \sqrt{4\cdot\frac{3}{4}} = 2\sqrt{\frac{3}{4}} = 2(1-\frac{1}{4})^{1/2}$. *Wait – better:* $\sqrt{3} = \sqrt{4(1-\frac{1}{4})} = 2(1-\frac{1}{4})^{1/2}$*?*

No. $3 = 4 \cdot \frac{3}{4}$ *is not helpful. Instead: use the expansion directly. Set* $x = 2$*? No,* $|x| < 1$ *required. Correct approach:* $\sqrt{3} = \sqrt{1+2}$*, but* $|x| = 2 > 1$ *– invalid. Better:* $\sqrt{3} = \sqrt{4-1} = 2\sqrt{1-\frac{1}{4}} = 2\left(1-\frac{1}{4}\right)^{1/2}$*. Substitute* $x = -\frac{1}{4}$*:* $2\left(1-\frac{1}{8}-\frac{1}{128}\right) = 2\cdot\frac{111}{128} = \frac{111}{64}$*. Check:* $\frac{111}{64} \approx 1.734$ *vs* $\sqrt{3} \approx 1.732$ *– good approximation. The error: substituting* $x = 2$ *directly (outside the validity range) or not rewriting* $\sqrt{3}$ *into the form* $(1+x)^{1/2}$ *with* $|x| < 1$*.*

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

Apply S_∞ formula	$\frac{8}{1-r} = 24 \Rightarrow 1 - r = \frac{1}{3}$	**M1**
Solve	$r = \frac{2}{3}$	**A1**
Check convergence	$\lvert r \rvert = \frac{2}{3} < 1$ ✓	**R1**

Final answer: $r = \frac{2}{3}$

Solution – Question 2

Identify GP	$u_1 = 2x - 1, \quad r = 2x - 1$	**M1**
Convergence condition	$\lvert 2x - 1 \rvert < 1 \Rightarrow 0 < x < 1$	**A1**
Sum formula	$S_\infty = \frac{2x-1}{1-(2x-1)} = \frac{2x-1}{2(1-x)}$	**M1**

Final answer: $S_\infty = \frac{2x-1}{2(1-x)}, \quad 0 < x < 1$

Solution – Question 3

Identify term	$k = 3 : \binom{6}{3}(2)^{6-3}(3x)^3$	**M1**
Compute	$20 \times 8 \times 27x^3$	**A1**
Coefficient	4320	**A1**

Final answer: Coefficient of $x^3 = 4320$

Solution – Question 4

(a) Terms 0,1,2	$1+\binom{8}{1}\frac{x}{2}+\binom{8}{2}\left(\frac{x}{2}\right)^2$	**M1**
Simplify	$1+4x+28\cdot\frac{x^2}{4}=1+4x+7x^2$	**A1**
(b) Set $x/2=0.005$, so $x=0.01$	$1+4(0.01)+7(0.01)^2=1+0.04+0.0007$	**M1**
Evaluate	1.04070	**A1**

Final answer: (a) $1+4x+7x^2+\cdots$ (b) $(1.005)^8\approx 1.04070$

Solution – Question 5

Write as $(1+u)^n$	$n=-\frac{1}{2},\quad u=2x$	**M1**
Term 1 and Term 2	$1+\left(-\frac{1}{2}\right)(2x)=1-x$	**A1**
Term 3	$\frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(2x)^2=\frac{3}{8}\cdot 4x^2=\frac{3x^2}{2}$	**A1**
Validity	$\lvert 2x\rvert<1\ \Rightarrow\ \lvert x\rvert<\frac{1}{2}$	**R1**

Final answer: $(1+2x)^{-1/2}\approx 1-x+\frac{3x^2}{2},\quad \lvert x\rvert<\frac{1}{2}$

Solution – Question 6

(a) $(1+x)^{1/2}$, $n=\frac{1}{2}$	$1+\frac{x}{2}+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}x^2=1+\frac{x}{2}-\frac{x^2}{8}$	**M1**
Validity	$\lvert x\rvert<1$	**A1**
(b) Write $\sqrt{3}=2(1-\frac{1}{4})^{1/2}$; set $x=-\frac{1}{4}$	$2\left(1+\frac{-1/4}{2}-\frac{(-1/4)^2}{8}\right)=2\left(1-\frac{1}{8}-\frac{1}{128}\right)$	**M1**
Simplify	$2\times\frac{128-16-1}{128}=2\times\frac{111}{128}=\frac{111}{64}$	**A1**

Final answer: (a) $1+\frac{x}{2}-\frac{x^2}{8}$, $\lvert x\rvert<1$ (b) $\sqrt{3}\approx\frac{111}{64}$
