Mathematical Induction

Mistake Analysis – Set I

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 1 – Proof
Level	Easy → Medium
Questions	6
Total marks	34
Instructions	Every induction proof must include all four steps shown below. No marks are awarded for verifying examples alone. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

THE FOUR STEPS OF MATHEMATICAL INDUCTION

Step 1 – Base case: verify the statement $P(n)$ for the smallest value of n.

Step 2 – Inductive hypothesis: assume $P(k)$ is true for some $k \geq n_0$ (write this out explicitly).

Step 3 – Inductive step: using the hypothesis, prove $P(k+1)$ is true. This is where all the algebra lives.

Step 4 – Conclusion: state that $P(n)$ is true for all $n \geq n_0$ by the principle of mathematical induction.

The conclusion must reference both the base case and the inductive step. Without it, no R1 mark.

Question 1 Easy [5 marks]

Prove by mathematical induction that for all positive integers n:

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}.$$

MISTAKE ANALYSIS

Base case *$(n=1)$: LHS $= 1$. RHS $= \frac{1\cdot 2}{2} = 1$. ✓*

Inductive hypothesis: *assume $\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$ for some $k \geq 1$.*

Inductive step: $\sum_{r=1}^{k+1} r = \frac{k(k+1)}{2} + (k+1) = (k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$. *This is the formula with $n = k+1$. ✓*

The critical error: *students who write "assume* $\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$*, therefore* $\sum_{r=1}^{k+1} r = \frac{(k+1)(k+2)}{2}$*" (jumping directly to the conclusion without algebraic justification) earn no marks for the inductive step. The step must show how the* $k+1$ *case follows from the* k *case by adding the* $(k+1)$*th term to both sides of the hypothesis.*

Question 2 Easy [5 marks]

Prove by mathematical induction that for all positive integers n:

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}.$$

MISTAKE ANALYSIS

Base case *($n = 1$): LHS* $= 1$*. RHS* $= \frac{1\cdot 2\cdot 3}{6} = 1$*. ✓*

Inductive step: $\displaystyle\sum_{r=1}^{k+1} r^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2.$

Factor $(k+1)$*:* $= \frac{(k+1)[k(2k+1)+6(k+1)]}{6} = \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$.

This is the formula with $n = k+1$ *(since* $2(k+1)+1 = 2k+3$*). ✓*

The algebraic trap: *students who factor incorrectly at the step* $\frac{k(k+1)(2k+1)}{6} + (k+1)^2$ *– typically by failing to find the common factor of* $(k+1)$ *and then expanding poorly – reach a result that does not match the formula. Always factor out* $(k+1)$ *immediately after adding the new term.*

Question 3 Easy–Medium [5 marks]

Prove by mathematical induction that $3^n - 1$ is divisible by 2 for all positive integers n.

MISTAKE ANALYSIS

Base case *($n = 1$):* $3^1 - 1 = 2$*, divisible by 2. ✓*

Inductive hypothesis: *assume* $3^k - 1 = 2m$ *for some integer* m*.*

Inductive step: $3^{k+1}-1=3\cdot 3^k-1=3(3^k-1)+3-1=3(2m)+2=2(3m+1)$. *Since* $3m+1$ *is an integer,* $3^{k+1}-1$ *is divisible by 2.* ✓

The structural error: *students who write "*$3^{k+1}-1=3\cdot 3^k-1$*, which is divisible by 2 because* 3^k *is divisible by 2 by the inductive hypothesis" are applying the hypothesis incorrectly. The hypothesis states* 3^k-1 *is divisible by 2, not* 3^k*. The correct approach: write* $3^{k+1}-1$ *in terms of* 3^k-1 *by adding and subtracting 1.*

Question 4 Medium [6 marks]

Prove by mathematical induction that for all integers $n \geq 1$ and $r \neq 1$:

$$\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}.$$

MISTAKE ANALYSIS

Base case *($n=1$): LHS* $=1+r$*. RHS* $=\frac{r^2-1}{r-1}=r+1$*.* ✓

Wait – base case should be $n=0$ *(empty sum if starting from* $j=0$*)... actually* $n=1$*: LHS* $=r^0+r^1=1+r$*, RHS* $=\frac{r^2-1}{r-1}=1+r$*.* ✓

Inductive step: $\sum_{j=0}^{k+1} r^j = \frac{r^{k+1}-1}{r-1}+r^{k+1}=\frac{r^{k+1}-1+r^{k+1}(r-1)}{r-1}=\frac{r^{k+2}-1}{r-1}$*. This is the formula with* $n=k+1$*.* ✓

The sign error: *when combining fractions in the inductive step,* $\frac{r^{k+1}-1}{r-1}+r^{k+1}$*, students often write the numerator as* $r^{k+1}-1+r^{k+1}=2r^{k+1}-1$ *(forgetting the factor* $(r-1)$ *when putting over a common denominator). The correct numerator is* $r^{k+1}-1+r^{k+1}(r-1)=r^{k+2}-1$*.*

Question 5 Medium [6 marks]

Prove by mathematical induction that $2^n > n$ for all positive integers n.

MISTAKE ANALYSIS

Base case *($n = 1$):* $2^1 = 2 > 1$. ✓

Inductive hypothesis: *assume* $2^k > k$ *for some* $k \geq 1$.

Inductive step: $2^{k+1} = 2 \cdot 2^k > 2k = k + k \geq k + 1$ *(since* $k \geq 1$*). So* $2^{k+1} > k + 1$. ✓

The missing step: *students who write* $2^{k+1} = 2 \cdot 2^k > 2k > k + 1$ *without justifying why* $2k > k + 1$ *lose the R1 mark. The step* $2k \geq k + 1$ *requires stating* $k \geq 1$*, which gives* $k \geq 1$*, so* $k + k \geq k + 1$*. In inequality inductions, every inequality step must be explicitly justified. "$>$" without justification is not a proof.*

Question 6 Medium [7 marks]

Prove by mathematical induction that $n^3 - n$ is divisible by 6 for all positive integers n.

MISTAKE ANALYSIS

Base case *($n = 1$):* $1 - 1 = 0$*, divisible by 6.* ✓

Inductive hypothesis: *assume* $k^3 - k = 6m$ *for some integer* m.

Inductive step: $(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1 = k^3 - k + 3k^2 + 3k = (k^3 - k) + 3k(k+1)$. *By IH:* $k^3 - k = 6m$*. Also* $k(k+1)$ *is the product of two consecutive integers, so it is divisible by 2:* $k(k+1) = 2t$ *for some integer* t*. Thus* $(k+1)^3 - (k+1) = 6m + 3 \cdot 2t = 6m + 6t = 6(m + t)$*, divisible by 6.* ✓

The incomplete argument: *students who write "$3k(k+1)$ is divisible by 6 because one of k or $k+1$ is even" earn the mark only if they identify which is even and why. Alternatively, note $k(k+1)$ is always even (consecutive integers). In divisibility proofs, the argument must be airtight – "one of them is even" is acceptable if stated clearly as a general fact about consecutive integers.*

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark. Every proof needs: base case, inductive hypothesis, inductive step, conclusion.

Solution – Question 1

Base case $(n = 1)$: LHS $= 1 =$ RHS $= \frac{1\cdot 2}{2}$. ✓

Inductive hypothesis: assume $\sum_{r=1}^{k} r = \frac{k(k+1)}{2}$.

Inductive step:

Add $(k+1)$th term: $\frac{k(k+1)}{2} + (k+1)$ $(k+1)\left(\frac{k}{2}+1\right) = \frac{(k+1)(k+2)}{2}$ **M1**

Conclusion: True for $n = 1$; if true for $n = k$ then true for $n = k + 1$. By induction, true for all $n \geq 1$. R1

Solution – Question 2

Base case $(n = 1)$: $1 = \frac{1\cdot 2\cdot 3}{6} = 1$. ✓

Inductive step:

$\frac{k(k+1)(2k+1)}{6} + (k+1)^2$ $= \frac{(k+1)(2k^2+7k+6)}{6}$ **M1**

$= \frac{(k+1)(k+2)(2k+3)}{6}$ Formula for $n = k + 1$ ✓ **A1**

Conclusion: True for all $n \geq 1$ by induction. R1

Solution – Question 3

Base case $(n = 1)$: $3 - 1 = 2 = 2 \cdot 1$. ✓

Inductive step: Let $3^k - 1 = 2m$.

$3^{k+1} - 1 = 3(3^k - 1) + 2 = 3(2m) + 2$ $2(3m+1)$ **M1**

Conclusion: $3m+1 \in \mathbb{Z}$, so $2 \mid 3^{k+1}-1$. True for all $n \geq 1$ by induction. R1

Solution – Question 4

Base case $(n=1)$: $1+r = \frac{r^2-1}{r-1} = r+1$. ✓

Inductive step:

$\frac{r^{k+1}-1}{r-1} + r^{k+1}$	$= \dfrac{r^{k+2}-1}{r-1}$	**M1**

Conclusion: Formula holds for $n = k+1$. True for all $n \geq 1$ by induction. R1

Solution – Question 5

Base case $(n=1)$: $2 > 1$. ✓

Inductive step: assume $2^k > k$.

$2^{k+1} = 2 \cdot 2^k > 2k = k+k \geq k+1$	(since $k \geq 1$)	**M1**

Conclusion: $2^{k+1} > k+1$. True for all $n \geq 1$ by induction. The step $k+k \geq k+1$ requires $k \geq 1$. R1

Solution – Question 6

Base case $(n=1)$: $0 = 6 \cdot 0$. ✓

Inductive step: let $k^3 - k = 6m$.

$(k+1)^3 - (k+1) = (k^3-k) + 3k(k+1)$	$= 6m + 3k(k+1)$	**M1**
$k(k+1)$ is product of consecutive integers: $k(k+1) = 2t$		**R1**
Hence	$6m + 6t = 6(m+t)$ ✓	**A1**

Conclusion: True for all $n \geq 1$ by induction. R1
