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Direct Proof & Proof by Contradiction

Mistake Analysis – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 1 – Proof
Level	Medium → Hard
Questions	6
Marks	35 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Direct proof: start from the hypothesis and derive the conclusion through a chain of valid logical steps.

Proof by contradiction: assume the negation of what you want to prove. Derive a logical contradiction. Conclude that the assumption must be false, so the original statement is true.

In every proof: state clearly what you are assuming, what you are proving, and how the steps connect. A sequence of true statements is not a proof unless the logical flow is explicit.

Question 1 — Medium — [5 marks]

Prove directly that the sum of two odd integers is even.

MISTAKE ANALYSIS

Let $m = 2a+1$ and $n = 2b+1$ for integers a, b. $m+n = 2a+1+2b+1 = 2(a+b+1)$. Since $a+b+1$ is an integer, $m+n$ is even. ✓ The error: students who write "odd + odd = even" as a conclusion without algebraic justification earn no marks. The proof requires expressing odd integers in the form $2k+1$ and showing the sum has the form $2(\text{integer})$. Always begin by writing the general form of each type of integer.

Question 2 — Medium — [5 marks]

Prove directly that if n is odd, then n^2 is odd.

MISTAKE ANALYSIS

Let $n = 2k + 1$ for some integer k. $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $2k^2 + 2k$ is an integer, n^2 is odd. ✓ Students who write "n is odd, so n^2 is odd because multiplying odd by odd gives odd" are making a circular appeal to the very fact they are asked to prove. The proof must use the algebraic definition $n = 2k + 1$ and derive the form $n^2 = 2(\text{integer}) + 1$.

Question 3 Medium [6 marks]

Prove by contradiction that $\sqrt{2}$ is irrational.

MISTAKE ANALYSIS

Assume *$\sqrt{2}$ is rational: $\sqrt{2} = \frac{p}{q}$ where p, q are integers with no common factor (i.e. the fraction is in lowest terms) and $q \neq 0$. Then $2 = \frac{p^2}{q^2}$, so $p^2 = 2q^2$. Thus p^2 is even, which means p is even (since if p were odd, p^2 would be odd – proved in Q2). Write $p = 2m$: then $4m^2 = 2q^2$, so $q^2 = 2m^2$, meaning q^2 is even, so q is even. But then p and q are both even, contradicting the assumption that $\frac{p}{q}$ is in lowest terms.* ***Contradiction.*** *So $\sqrt{2}$ is irrational. ✓ The most common error: students who write "assume $\sqrt{2} = p/q$" without specifying that p/q is in lowest terms cannot reach the contradiction. The "no common factor" assumption is essential – without it, p and q both being even is not a contradiction.*

Question 4 Medium–Hard [6 marks]

Prove by contradiction that there is no largest prime number.

MISTAKE ANALYSIS

Assume *there are finitely many primes: $p_1, p_2, \ldots, p_n$ (a complete list). Let $N = p_1 p_2 \cdots p_n + 1$. N is either prime or composite. If N is prime: it is larger than all p_i, contradicting the assumption that the list is complete. If N is composite: it has a prime factor p_j. But $p_j \mid p_1 p_2 \cdots p_n$, so $p_j \mid N - p_1 \cdots p_n = 1$. But no prime divides 1. Contradiction. In both cases we reach a contradiction. So there are infinitely many primes. ✓ Students who claim $N = p_1 p_2 \cdots p_n + 1$ is itself prime are wrong in general ($2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 =$*

$30031 = 59 \times 509$). The argument only requires that N has a prime factor not in the original list.

Question 5 Hard [6 marks]

Prove by contradiction that $\log_2 3$ is irrational.

MISTAKE ANALYSIS

Assume *$\log_2 3 = \frac{p}{q}$ where p, q are positive integers. Then $2^{p/q} = 3$, so $2^p = 3^q$. But 2^p is even and 3^q is odd for all positive integers p, q. An even number cannot equal an odd number.* ***Contradiction.*** *So $\log_2 3$ is irrational. ✓ Students who attempt this using the rational root theorem or direct computation miss the point. The key insight is parity: 2^p is always even; 3^q is always odd. These cannot be equal.*

Question 6 Hard [7 marks]

Prove that if n^2 is even then n is even (i.e. the converse of Question 2). Hence prove that $\sqrt{3}$ is irrational.

MISTAKE ANALYSIS

Part 1 *(contrapositive): the contrapositive of "n^2 even $\Rightarrow n$ even" is "n odd $\Rightarrow n^2$ odd", which was proved in Q2. Since a statement and its contrapositive are logically equivalent: if n^2 is even, then n is even. ✓*

Part 2 *($\sqrt{3}$ irrational): assume $\sqrt{3} = p/q$ in lowest terms. $3q^2 = p^2$. So $3 \mid p^2$. Since 3 is prime: $3 \mid p$, so $p = 3m$. $3q^2 = 9m^2 \Rightarrow q^2 = 3m^2 \Rightarrow 3 \mid q$. Both p and q divisible by 3: contradicts lowest terms. ✓*

The error in Part 1: students who prove "n even $\Rightarrow n^2$ even" have proved the converse of Q2, not the statement required here. Read the direction of implication carefully: here we need "n^2 even $\Rightarrow n$ even", proved via its contrapositive.

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

Let $m = 2a + 1$, $n = 2b + 1$	$m + n = 2(a + b + 1)$: even ✓	**M1**

Solution – Question 2

Let $n = 2k + 1$	$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$: odd ✓	**M1**

Solution – Question 3

Assume $\sqrt{2} = p/q$ in lowest terms	$p^2 = 2q^2 \Rightarrow p$ even $\Rightarrow p = 2m$	**M1**
	$4m^2 = 2q^2 \Rightarrow q^2 = 2m^2 \Rightarrow q$ even	**M1**
p and q both even: contradicts lowest terms	$\sqrt{2}$ is irrational ✓	**R1**

Solution – Question 4

Assume finitely many primes $p_1, \ldots, p_n$; $N = p_1 \cdots p_n + 1$		**M1**
Any prime factor of N divides $N - p_1 \cdots p_n = 1$: impossible		**R1**
	Infinitely many primes ✓	**A1**

Solution – Question 5

Assume $\log_2 3 = p/q$; $2^p = 3^q$	2^p even, 3^q odd: contradiction ✓	**M1**

Solution – Question 6

Part 1: contrapositive of "n^2 even $\Rightarrow n$ even is Q2 ✓		**R1**
Part 2: assume $\sqrt{3} = p/q$; $3 \mid p^2 \Rightarrow 3 \mid p \Rightarrow p = 3m$	$q^2 = 3m^2 \Rightarrow 3 \mid q$	**M1**
p, q both divisible by 3: contradicts lowest terms	$\sqrt{3}$ irrational ✓	**R1**

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