

Proof by Counterexample & Disproving Conjectures

Mistake Analysis – Set III

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 1 – Proof
Level	Medium → Hard
Questions	6
Marks	33 total. M1 method · A1 accuracy · R1 reasoning.

BEFORE YOU BEGIN

A **universal statement** claims something is true for *all* values in a set. Example: “For all integers n , $n^2 + n + 41$ is prime.”

To **disprove** a universal statement, it suffices to produce **one counterexample** – a single value for which the statement fails.

To **prove** a universal statement, a counterexample is never sufficient. Checking many cases does not constitute a proof.

Deciding which is needed: read whether the statement is asking for a proof (it is true for all n) or a disproof (find an n for which it fails). These require completely different responses.

Question 1

Medium

[4 marks]

Disprove the following conjecture by finding a counterexample: “For all integers $n \geq 1$, $n^2 + n + 41$ is prime.”

MISTAKE ANALYSIS

At $n = 40$: $40^2 + 40 + 41 = 1600 + 40 + 41 = 1681 = 41^2$. Not prime. ✓ (Also works at $n = 41$: $41^2 + 41 + 41 = 41(41 + 1 + 1) = 41 \times 43$.) Students who test $n = 1, 2, 3, \dots$ and find primes in each case conclude (incorrectly) that the statement is proved. No finite number of examples proves a universal statement. One counterexample disproves it – but you must find one that actually fails. The formula $n^2 + n + 41$ is prime for $n = 0, 1, 2, \dots, 39$ (Euler’s lucky number), so small cases do not reveal the counterexample.

Question 2

Medium

[5 marks]

Determine whether each statement is true or false. If true, prove it. If false, give a counterexample.

- (a) “For all real x , $x^2 \geq x$.”
(b) “For all real $x \neq 0$, $x^2 \geq x$.”

MISTAKE ANALYSIS

(a) **False.** Counterexample: $x = 0.5$. $0.5^2 = 0.25 < 0.5$. (b) **False.** Same counterexample works: $x = 0.5 \neq 0$, $0.25 < 0.5$. (Or any $x \in (0, 1)$.) Students who attempt to prove (a) by squaring both sides or algebraic manipulation often produce a chain of inequalities that contains an error. The claim is simply false – check a small positive fraction first before attempting a proof. The lesson: before writing a proof, test a few values. If any value fails, you have a counterexample and the statement is false.

Question 3

Medium

[5 marks]

Disprove: “The product of two irrational numbers is irrational.” Then determine whether the *sum* of two irrational numbers is always irrational.

MISTAKE ANALYSIS

Product: Counterexample: $\sqrt{2} \times \sqrt{2} = 2$, which is rational. ✓ (Also: $\sqrt{2} \times \sqrt{8} = \sqrt{16} = 4$.) **Sum:** Also false. Counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$, which is rational. Students who think “irrational \times irrational is always irrational” by analogy with “rational \times rational is rational” are wrong. The closure properties of irrationals under addition and multiplication both fail. Only one counterexample is needed to disprove each claim.

Question 4

Medium–Hard

[6 marks]

Consider the statement: “For all positive integers n , if n^2 is divisible by 4, then n is divisible by 4.”

- (a) Disprove the statement.

- (b) State the correct relationship between divisibility of n^2 by 4 and divisibility of n by 4, and prove it.

MISTAKE ANALYSIS

(a) *Counterexample: $n = 2$. $n^2 = 4$, divisible by 4. But $n = 2$ is not divisible by 4. ✓* (b) *The correct statement: n^2 divisible by 4 iff n is divisible by 2 (i.e. n is even). Proof: if $n = 2k$ then $n^2 = 4k^2$, divisible by 4. Conversely: if n^2 divisible by 4 then n^2 is even, so n is even (by Q6 from Set II). Students who confuse “divisible by 4” with “divisible by 2” produce an incorrect correction. The key insight: $n^2 \equiv 0 \pmod{4}$ does not force $n \equiv 0 \pmod{4}$; it only forces $n \equiv 0 \pmod{2}$.*

Question 5

Hard

[6 marks]

Determine whether the following statements are true or false. Prove or disprove each.

- (a) “For all real a, b : if $a^2 = b^2$ then $a = b$.”
(b) “For all real a, b : if $a^2 = b^2$ then $a = \pm b$.”

MISTAKE ANALYSIS

(a) **False.** *Counterexample: $a = 1, b = -1$. $1^2 = (-1)^2 = 1$ but $1 \neq -1$.* (b) **True.** $a^2 = b^2 \Rightarrow a^2 - b^2 = 0 \Rightarrow (a - b)(a + b) = 0 \Rightarrow a = b$ or $a = -b$, i.e. $a = \pm b$. ✓ *Students who attempt to prove (a) by taking square roots on both sides write $\sqrt{a^2} = \sqrt{b^2}$, giving $|a| = |b|$, which is the correct result – but then incorrectly conclude $a = b$ rather than $|a| = |b|$ (i.e. $a = \pm b$). $\sqrt{x^2} = |x|$, not x .*

Question 6

Hard

[7 marks]

- (a) Give a counterexample to: “For all functions f and g , if $f \circ g = g \circ f$ then $f = g$.”
(b) Give a counterexample to: “For all $n \in \mathbb{Z}^+$, $2^n - 1$ is prime.”
(c) The statement “ $P(n)$ is true for $n = 1, 2, 3, \dots, 1000$ ” does not prove $P(n)$ is true for all n . Explain why, and give an example of a statement that is true for many values but false in general.

MISTAKE ANALYSIS

(a) Let $f(x) = x + 1$ and $g(x) = x$. Then $f \circ g(x) = x + 1 = g \circ f(x)$, but $f \neq g$. Or simpler: $f = g =$ any constant function. (b) $n = 4$: $2^4 - 1 = 15 = 3 \times 5$. Not prime. $\checkmark (2^{11} - 1 = 2047 = 23 \times 89$ is another example.) (c) Proof by example fails because the domain is infinite. Even verifying 1000 cases leaves infinitely many unchecked. Example: $n^2 + n + 41$ is prime for $n = 0, \dots, 39$ but composite at $n = 40$. Students who think 1000 verified cases constitute a “near-proof” or give “strong evidence” are making a logical error. A universal statement is either proved for all cases or it is not proved. There is no middle ground in mathematics.

WORKED SOLUTIONS – SET III – COUNTEREXAMPLE & DISPROVING

M1 method · A1 accuracy · R1 reasoning

Solution – Question 1

$n = 40$: $40^2 + 40 + 41 = 1681 = 41^2$ *Not prime : statement disproved* ✓ **A1**

Solution – Question 2

(a) $x = 0.5$: *False* ✓ **A1**
 $0.25 < 0.5$

(b) Same counterexample ($x = 0.5 \neq 0$) *False* ✓ **A1**

Solution – Question 3

Product: $\sqrt{2} \times (-\sqrt{2}) = -2 \notin \mathbb{Q}$ *Statement disproved* ✓ **A1**

Sum: $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$ *Also false* ✓ **A1**

Solution – Question 4

(a) $n = 2$: 4 div *Disproved* ✓ **A1**
by 4, but 2 not div by 4

(b) $n = 2k \Rightarrow n^2 = 4k^2$, div by 4; converse: n^2 div by 4 $\Rightarrow n^2$ even $\Rightarrow n$ even **M1**
Correct statement: n^2 div by 4 iff n is even **R1**

Solution – Question 5

- (a) $a = 1, b = -1$: $1 = 1$ but $1 \neq -1$ *False* ✓ **A1**
- (b) $a^2 - b^2 = (a - b)(a + b) = 0 \Rightarrow a = b$ or $a = -b$ ✓ **M1**
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Solution – Question 6

- (a) $f(x) = x + 1, g(x) = x$: $f \circ g = g \circ f = x + 1$, but $f \neq g$ **A1**
- (b) $n = 4$: $2^4 - 1 = 15 = 3 \times 5$; not prime **A1**
- (c) Domain is infinite; 1000 cases leave infinitely many unchecked; $n^2 + n + 41$ is the canonical example **R1**
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