THE SANCTUARY OF LEARNING – ABDULWADOOD.ORG

Continuous Distributions

Mistake Analysis – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 4 – Statistics & Probability
Level	Medium → Hard
Questions	6
Total marks	33
GDC	You may use your GDC. Show all working and state any GDC inputs used.
Mark scheme	**M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

BEFORE YOU BEGIN

If $X \sim N(\mu, \sigma^2)$, then $Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$.

$P(X < a) = P\left(Z < \dfrac{a-\mu}{\sigma}\right)$.

For the **inverse normal**: given $P(X < a) = p$, find $a = \mu + z_p\sigma$ where $z_p = \Phi^{-1}(p)$.

Central Limit Theorem: if $X_1, \dots, X_n$ are i.i.d. with mean μ and variance σ^2, then $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ approximately for large n.

Question 1 — Medium — [4 marks]

$X \sim N(50,\ 16)$. Find:

(a) $P(X < 54)$

(b) $P(46 < X < 58)$

Give answers correct to 4 decimal places.

MISTAKE ANALYSIS

Note: $X \sim N(50, 16)$ *means* $\mu = 50$ *and* $\sigma^2 = 16$, *so* $\sigma = 4$ *(not* $\sigma = 16$*). (a)* $z = \dfrac{54-50}{4} = 1$. $P(X < 54) = P(Z < 1) \approx 0.8413$. *GDC:* `normalcdf(-99,54,50,4)` ≈ 0.8413. *(b)* $P(46 < X < 58) = P(-1 < Z < 2) \approx 0.8186$. *GDC:* `normalcdf(46,58,50,4)` ≈ 0.8186. *The most common error: treating the second parameter as* σ *(standard deviation) rather than* σ^2 *(variance).* $N(50,16)$ *has standard deviation* $\sigma = 4$*, not 16. Always identify* $\sigma = \sqrt{\sigma^2}$ *before computing any* z*-score.*

Question 2 Medium [6 marks]

$X \sim N(\mu, \sigma^2)$. Given that $P(X < 10) = 0.2$ and $P(X < 20) = 0.8$, find μ and σ.

MISTAKE ANALYSIS

Since $P(X < 10) = 0.2$ and $P(X < 20) = 0.8 = 1 - 0.2$, the distribution is symmetric about the midpoint: $\mu = \frac{10+20}{2} = 15$. From $P(X < 10) = 0.2$: $z_{0.2} = \Phi^{-1}(0.2) \approx -0.8416$. $\frac{10-15}{\sigma} = -0.8416 \Rightarrow \sigma = \frac{5}{0.8416} \approx 5.94$. GDC: `invNorm(0.2,0,1)`$\approx -0.8416$. Students who set up two equations $\frac{10-\mu}{\sigma} = z_1$ and $\frac{20-\mu}{\sigma} = z_2$ without first exploiting the symmetry ($z_1 = -z_2$ when $p_1 + p_2 = 1$) make unnecessary algebraic work. Spot the symmetry: $0.2 + 0.8 = 1$ means the two values are equidistant from μ.

Question 3 Medium [5 marks]

$X \sim N(100, 225)$. Find the value of k such that $P(X > k) = 0.1$. Give your answer correct to 3 significant figures.

MISTAKE ANALYSIS

$\sigma^2 = 225$, so $\sigma = 15$. $P(X > k) = 0.1 \Rightarrow P(X < k) = 0.9$. $k = \mu + z_{0.9}\sigma = 100 + 1.2816 \times 15 \approx 100 + 19.22 = 119$ (3 s.f.). GDC: `invNorm(0.9,100,15)`≈ 119. The error: students who write $P(X > k) = 0.1$ and then look up $z_{0.1} = -1.2816$ (the left-tail value) get $k = 100 + (-1.2816)(15) \approx 80.8$, which is the value with $P(X < k) = 0.1$, not $P(X > k) = 0.1$. Convert right-tail to left-tail first: $P(X > k) = 0.1 \Rightarrow P(X < k) = 0.9$.

Question 4 Medium–Hard [6 marks]

The lengths of components produced by a machine are normally distributed with mean $\mu = 20\,\text{cm}$ and standard deviation $\sigma = 3\,\text{cm}$. A random sample of 36 components is taken. Find the probability that the sample mean $\bar{X}$ lies between 19 cm and 21 cm.

MISTAKE ANALYSIS

By the Central Limit Theorem: $\bar{X} \sim N\left(20, \frac{9}{36}\right) = N(20, 0.25)$. *Standard error:* $\sigma_{\bar{X}} = \frac{3}{\sqrt{36}} = 0.5$. $P(19 < \bar{X} < 21) = P\left(\frac{19-20}{0.5} < Z < \frac{21-20}{0.5}\right) = P(-2 < Z < 2) \approx 0.9545$. *GDC:* `normalcdf(19,21,20,0.5)`$\approx$ 0.9545. *The critical error: using* $\sigma = 3$ *(population s.d.) instead of* $\sigma/\sqrt{n} = 0.5$ *(standard error of the mean). The distribution of* $\bar{X}$ *has standard deviation* $\sigma/\sqrt{n}$, *not* σ. *Dividing by* $\sqrt{n}$ *is mandatory.*

Question 5 Hard [6 marks]

$X \sim N(30, 64)$. Find the value of a such that $P(X < a) = 0.75$. Give your answer correct to 3 significant figures.

MISTAKE ANALYSIS

$\sigma^2 = 64$, *so* $\sigma = 8$. $z_{0.75} = \Phi^{-1}(0.75) \approx 0.6745$. $a = \mu + z_{0.75}\sigma = 30 + 0.6745 \times 8 \approx 30 + 5.40 = 35.4$ *(3 s.f.). GDC:* `invNorm(0.75,30,8)`$\approx$ 35.4. *Students who use* $\sigma = 64$ *(variance instead of standard deviation) compute* $a = 30 + 0.6745 \times 64 \approx 73.2$ *– a value far outside the plausible range. Always extract* $\sigma = \sqrt{\sigma^2}$ *from the* $N(\mu, \sigma^2)$ *notation before using the inverse normal. A quick sanity check:* a *should be within 2 or 3 standard deviations of* μ.

Question 6 Hard [6 marks]

The scores on an examination are normally distributed with mean 75 and standard deviation 10. Two students are chosen at random, independently.

(a) Find the probability that a single student scores more than 85.

(b) Find the probability that at least one of the two students scores more than 85.

MISTAKE ANALYSIS

(a) $P(X > 85) = P(Z > 1) = 1 - \Phi(1) \approx 1 - 0.8413 = 0.1587$. *GDC:* `normalcdf(85,99,75,10)`$\approx$ 0.1587. *(b) Let* $p = P(X > 85) \approx 0.1587$. $P(\textit{at least one} > 85) = 1 - P(\textit{neither} > 85) = 1 - (1-p)^2$. $= 1 - (0.8413)^2 \approx 1 - 0.7078 = 0.2922$. *The error: students who write* $P(\textit{at least one}) = 2p = 2(0.1587) = 0.3174$ *are double-counting the event that both score above 85. Use the complement:* $1 - (1-p)^2$, *not* $2p$.

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark. GDC inputs shown in `typewriter font`.

Solution – Question 1

$\sigma = \sqrt{16} = 4$; 0.8413 **M1**
(a) $z = \frac{54-50}{4} = 1$; GDC: `normalcdf(-99,54,50,4)`
(b) GDC: `normalcdf(46,58,50,4)` 0.8186 **A1**

Final answer: (a) 0.8413 (b) 0.8186

Solution – Question 2

$P(X < 10) + P(X < 20) = 1$: symmetric, so $\mu = \frac{10+20}{2} = 15$ **R1**

$\frac{10-15}{\sigma} = \Phi^{-1}(0.2) \approx -0.8416$; GDC: `invNorm(0.2,0,1)` $\sigma = \dfrac{5}{0.8416} \approx 5.94$ **M1**

Final answer: $\mu = 15$; $\sigma \approx 5.94$

Solution – Question 3

$\sigma = 15$; $P(X > k) = 0.1 \Rightarrow P(X < k) = 0.9$ **M1**

GDC: `invNorm(0.9,100,15)` $k \approx 119$ **A1**

Final answer: $k \approx 119$

Solution – Question 4

$\bar{X} \sim N\left(20, \frac{9}{36}\right)$; **M1**
$\sigma_{\bar{X}} = \frac{3}{\sqrt{36}} = 0.5$
GDC: 0.9545 **A1**
`normalcdf(19,21,20,0.5)`

Final answer: $P(19 < \bar{X} < 21) \approx 0.9545$

Solution – Question 5

$\sigma = \sqrt{64} = 8$; $a \approx 35.4$ **M1**
GDC:
`invNorm(0.75,30,8)`

Final answer: $a \approx 35.4$

Solution – Question 6

(a) $P(X > 85) = 1 - \Phi(1)$; $p \approx 0.1587$ **M1**
GDC:
`normalcdf(85,99,75,10)`
(b) 0.2922 **A1**
$P(\text{at least one}) = 1 - (1-p)^2 = 1 - (0.8413)^2$

Final answer: (a) 0.1587 (b) 0.2922
