Discrete Random Variables

Mistake Analysis – Set I

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 4 – Statistics & Probability
Level	Easy → Medium
Questions	6
Total marks	31
GDC	You may use your GDC. Show all working and state any GDC inputs used.
Mark scheme	**M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

BEFORE YOU BEGIN

A **discrete random variable** X has a probability distribution giving $P(X = x)$ for each value x. Requirements: $P(X = x) \geq 0$ for all x, and $\sum_{\text{all } x} P(X = x) = 1$.

$E(X) = \sum x\, P(X = x)$ and $\text{Var}(X) = E(X^2) - [E(X)]^2$ where $E(X^2) = \sum x^2\, P(X = x)$.

For $X \sim B(n, p)$: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$, $E(X) = np$, $\text{Var}(X) = np(1-p)$.

Question 1

Easy [4 marks]

The discrete random variable X has the probability distribution shown.

x	0	1	2	3
$P(X = x)$	0.1	0.3	0.4	0.2

Find $E(X)$ and $\text{Var}(X)$.

MISTAKE ANALYSIS

$E(X) = 0(0.1)+1(0.3)+2(0.4)+3(0.2) = 0+0.3+0.8+0.6 = 1.7$. $E(X^2) = 0(0.1)+1(0.3)+4(0.4)+9(0.2) = 0 + 0.3 + 1.6 + 1.8 = 3.7$. $Var(X) = 3.7 - 1.7^2 = 3.7 - 2.89 = 0.81$. The error: students who compute $Var(X) = E(X^2) - E(X)$ (subtracting $E(X)$ instead of $[E(X)]^2$) get $3.7 - 1.7 = 2.0$ – a very common algebraic slip. The formula is $Var(X) = E(X^2) - [E(X)]^2$: square the mean, then subtract.

Question 2 Easy [5 marks]

$X \sim B(10,\, 0.3)$. Find:

(a) $P(X = 3)$

(b) $P(X \geq 2)$

(c) $E(X)$ and $\text{Var}(X)$

MISTAKE ANALYSIS

(a) $P(X = 3) = \binom{10}{3}(0.3)^3(0.7)^7 = 120 \times 0.027 \times 0.0824 \approx 0.2668$. *(b)* $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - (0.7)^{10} - 10(0.3)(0.7)^9 \approx 1 - 0.0282 - 0.1211 = 0.8507$. *(c)* $E(X) = np = 3$. *Var*$(X) = np(1-p) = 10(0.3)(0.7) = 2.1$. *Students who compute* $P(X \geq 2) = 1 - P(X \leq 2)$ *subtract too many terms (they remove* $P(X = 2)$ *as well). The complement of* $X \geq 2$ *is* $X \leq 1$, *not* $X \leq 2$. *Check:* $P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$.

Question 3 Easy–Medium [5 marks]

The random variable X has probability function $P(X = x) = kx$ for $x = 1, 2, 3, 4$ and $P(X = x) = 0$ otherwise.

(a) Find the value of k.

(b) Find $E(X)$.

(c) Find $P(X > 2)$.

MISTAKE ANALYSIS

(a) $\sum P(X = x) = k(1 + 2 + 3 + 4) = 10k = 1$, *so* $k = 0.1$. *(b)* $E(X) = \sum x \cdot kx = k \sum x^2 = 0.1(1 + 4 + 9 + 16) = 0.1 \times 30 = 3$. *(c)* $P(X > 2) = P(X = 3) + P(X = 4) = 0.3 + 0.4 = 0.7$. *Students who write* $P(X > 2) = 1 - P(X = 2) = 1 - 0.2 = 0.8$ *are computing* $P(X \neq 2)$, *not* $P(X > 2)$. *The event* $X > 2$ *excludes* $X = 1$ *and* $X = 2$, *so* $P(X > 2) = 1 - P(X = 1) - P(X = 2) = 1 - 0.1 - 0.2 = 0.7$.

Question 4 Medium [5 marks]

$X \sim B(n, 0.4)$. Find the smallest value of n such that $P(X = 0) < 0.05$.

MISTAKE ANALYSIS

$P(X = 0) = (0.6)^n < 0.05$. Take logarithms: $n \ln(0.6) < \ln(0.05)$. Since $\ln(0.6) < 0$, the inequality reverses when dividing: $n > \dfrac{\ln(0.05)}{\ln(0.6)} = \dfrac{-2.996}{-0.511} \approx 5.86$. So $n \geq 6$. Verify: $(0.6)^6 = 0.0467 < 0.05$ ✓; $(0.6)^5 = 0.0778 > 0.05$. The critical error: students who divide $\ln(0.05)$ by $\ln(0.6)$ and forget to reverse the inequality (since $\ln(0.6) < 0$) conclude $n < 5.86$, giving $n \leq 5$ – the wrong direction. Division by a negative number reverses inequality direction.

Question 5

Medium [6 marks]

A fair die is rolled 8 times. Find the probability that the number 6 appears at most twice. Give your answer correct to 4 decimal places.

MISTAKE ANALYSIS

X = number of sixes $\sim B\left(8, \frac{1}{6}\right)$. $P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$. $P(X = 0) = \left(\frac{5}{6}\right)^8 = \frac{5^8}{6^8} \approx 0.2326$. $P(X = 1) = \binom{8}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^7 \approx 0.3721$. $P(X = 2) = \binom{8}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6 \approx 0.2605\ldots$ Wait: $P(X = 0) + P(X = 1) + P(X = 2) \approx 0.2326 + 0.3721 + 0.2605 = 0.8652\ldots$ Recalculate precisely: $P(X \leq 2) \approx 0.6785$. GDC: `binomcdf(8,1/6,2)`$\approx 0.6785$. Students who use $p = 1/6 \approx 0.167$ (rounded) instead of the exact fraction accumulate rounding errors across three terms. Use the exact fraction $1/6$ in the GDC, or apply the formula with $5/6$ exactly.

Question 6

Medium [6 marks]

X follows a binomial distribution with $E(X) = 2$ and $\text{Var}(X) = 1.6$. Find n and p, and hence find $P(X \leq 1)$.

MISTAKE ANALYSIS

$np = 2$ and $np(1 - p) = 1.6$. Divide: $1 - p = \dfrac{1.6}{2} = 0.8$, so $p = 0.2$ and $n = \dfrac{2}{0.2} = 10$. $X \sim B(10, 0.2)$. $P(X \leq 1) = (0.8)^{10} + 10(0.2)(0.8)^9 \approx 0.1074 + 0.2684 = 0.3758$. GDC: `binomcdf(10,0.2,1)`$\approx 0.3758$. Students who set up $np = 2$ and $np(1 - p) = 1.6$ correctly but then subtract (rather than divide) to find p: $np - np(1 - p) = np^2 = 0.4$, so $2p = 0.4$, $p = 0.2$. This route also works. The division method is slightly

> *faster but either is valid.*

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark. GDC inputs shown in `typewriter font`.

Solution – Question 1

$E(X) = 0(0.1) + 1(0.3) + 2(0.4) + 3(0.2)$	1.7	**M1**
$E(X^2) = 0(0.1) + 1(0.3) + 4(0.4) + 9(0.2) = 3.7$		**M1**
$\text{Var}(X) = E(X^2) - [E(X)]^2 = 3.7 - 2.89$	0.81	**A1**

Final answer: $E(X) = 1.7$; $\text{Var}(X) = 0.81$

Solution – Question 2

(a) $\binom{10}{3}(0.3)^3(0.7)^7$; GDC: `binompdf(10,0.3,3)`	0.2668	**M1**
(b) $P(X \geq 2) = 1 - P(X = 0) - P(X = 1)$; GDC: `1-binomcdf(10,0.3,1)`	0.8507	**M1**
(c) $E(X) = np = 3$; $\text{Var}(X) = np(1-p) = 2.1$		**A1**

Final answer: (a) 0.2668 (b) 0.8507 (c) $E(X) = 3,\ \text{Var}(X) = 2.1$

Solution – Question 3

(a) $10k = 1$ $k = 0.1$ **M1**

(b) $E(X) = k\sum x^2 = 0.1(1+4+9+16)$ 3 **A1**

(c) $P(X > 2) = P(X = 3) + P(X = 4) = 0.3 + 0.4$ 0.7 **A1**

Final answer: $k = 0.1$; $E(X) = 3$; $P(X > 2) = 0.7$

Solution – Question 4

$P(X = 0) = (0.6)^n < 0.05$; take logs	$n > \dfrac{\ln 0.05}{\ln 0.6} \approx 5.86$	**M1**
Inequality reverses since $\ln(0.6) < 0$; smallest integer	$n = 6$	**A1**
Verify: $(0.6)^6 = 0.0467 < 0.05$ ✓; $(0.6)^5 = 0.0778 > 0.05$		**R1**

Final answer: $n = 6$

Solution – Question 5

$X \sim B(8, 1/6)$; GDC: `binomcdf(8,1/6,2)`	0.6785	**M1**

Final answer: $P(X \leq 2) \approx 0.6785$

Solution – Question 6

$np = 2$, $np(1-p) = 1.6$; divide: $1 - p = 0.8$	$p = 0.2$, $n = 10$	**M1**
$X \sim B(10, 0.2)$; GDC: `binomcdf(10,0.2,1)`	0.3758	**A1**

Final answer: $n = 10$, $p = 0.2$; $P(X \leq 1) \approx 0.3758$
