Transformations & Rational Functions

Mistake Analysis – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 2 – Functions
Level	Medium → Hard
Questions	6
Total marks	33
Instructions	No calculator unless stated. Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

BEFORE YOU BEGIN

Key transformations of $y = f(x)$: $y = f(x) + a$ (shift up by a), $y = f(x - a)$ (shift right by a), $y = af(x)$ (stretch vertically by a), $y = f(ax)$ (compress horizontally by a), $y = -f(x)$ (reflect in x-axis), $y = f(-x)$ (reflect in y-axis).

For a **rational function** $f(x) = \frac{p(x)}{q(x)}$: vertical asymptotes occur where $q(x) = 0$ (and $p(x) \neq 0$); horizontal asymptote: $y = \frac{\text{leading coeff of } p}{\text{leading coeff of } q}$ when $\deg p = \deg q$; oblique asymptote: perform polynomial long division when $\deg p = \deg q + 1$.

$y = |f(x)|$: reflect any part of the graph below the x-axis upward.
$y = f(|x|)$: discard the graph for $x < 0$; reflect the $x > 0$ portion in the y-axis.

Question 1 — Medium — [4 marks]

The graph of $y = f(x)$ passes through (1, 3) and has a horizontal asymptote $y = 0$. Describe fully the transformations that map $y = f(x)$ onto each of the following, and state what happens to the point (1, 3) and the asymptote under each transformation.

(a) $y = f(x - 2) + 3$

(b) $y = -2f(x)$

(c) $y = f(3x)$

MISTAKE ANALYSIS

(a) Shift right 2, shift up 3. Point: $(1, 3) \to (3, 6)$. Asymptote: $y = 0 \to y = 3$. (b) Reflect in x-axis, stretch vertically by factor 2. Point: $(1, 3) \to (1, -6)$. Asymptote: $y = 0 \to y = 0$. (c) Horizontal compression by factor 3. Point: $(1, 3) \to (1/3, 3)$. Asymptote: $y = 0 \to y = 0$ (unchanged). For $y = f(3x)$: the x-coordinate is divided by 3 (not multiplied). Students who map $(1, 3)$ to $(3, 3)$ for $y = f(3x)$ are scaling in the wrong direction. $f(3x)$ reaches the same y-value at $x/3$, so the point moves to the left.

Question 2 Medium [5 marks]

Let $f(x) = \dfrac{2x-1}{x+3}$.

(a) State the equations of the vertical and horizontal asymptotes.

(b) Find the x- and y-intercepts.

(c) Hence sketch the graph of f, showing all key features.

MISTAKE ANALYSIS

(a) Vertical asymptote: $x = -3$ (denominator zero). Horizontal asymptote: $y = 2$ (ratio of leading coefficients, since both numerator and denominator are degree 1). (b) x-intercept: $2x - 1 = 0 \Rightarrow x = \frac{1}{2}$. y-intercept: $f(0) = -\frac{1}{3}$. (c) The graph has two branches separated by $x = -3$, approaching $y = 2$ as $x \to \pm\infty$. Students who state the horizontal asymptote as $y = \frac{2}{1} = 2$ correctly, but then draw the graph crossing the asymptote somewhere: for this type of rational function (degree 1 over degree 1), the graph never crosses the horizontal asymptote. Also: the vertical asymptote is at $x = -3$, not $x = 3$. The sign matters.

Question 3 Medium [6 marks]

Let $f(x) = x^2 - 4$.

(a) Sketch $y = |f(x)|$, clearly indicating any points where the graph meets the x-axis.

(b) Sketch $y = f(|x|)$, clearly indicating any points where the graph meets the x-axis.

(c) State the range of each function.

MISTAKE ANALYSIS

$f(x) = x^2 - 4 = (x-2)(x+2)$. Zeros at $x = \pm 2$, minimum at $(0, -4)$. (a) $y = |f(x)|$: the parabola dips below the x-axis between $x = -2$ and $x = 2$. Reflect this portion upward: the segment becomes an upward arch touching the x-axis at $x = \pm 2$, with a local maximum at $(0, 4)$. Range: $[0, \infty)$. (b) $y = f(|x|)$: discard the graph for $x < 0$; reflect the $x \geq 0$ portion in the y-axis. The result is symmetric about the y-axis, touching the x-axis only at $x = \pm 2$ and having minimum $(0, -4)$. Range: $[-4, \infty)$. The error: students swap the two constructions. For $|f(x)|$: reflect y-values; for $f(|x|)$: reflect x-values.

Question 4 Medium–Hard [6 marks]

Sketch the graph of $f(x) = \dfrac{x^2 - 1}{x^2 - 4}$, clearly showing: all asymptotes, all intercepts, and the behaviour as $x \to \pm\infty$.

MISTAKE ANALYSIS

$f(x) = \dfrac{(x-1)(x+1)}{(x-2)(x+2)}$. Vertical asymptotes: $x = \pm 2$. Horizontal asymptote: $y = 1$ (leading coefficients both 1; $y \to 1$ as $x \to \pm\infty$). x-intercepts: $x = \pm 1$. y-intercept: $f(0) = \frac{-1}{-4} = \frac{1}{4}$. The graph has three branches: for $x < -2$, for $-2 < x < 2$, and for $x > 2$. The middle branch ($-2 < x < 2$) passes through $(-1, 0)$, $(0, 1/4)$, $(1, 0)$ and lies below $y = 1$. Students who state the horizontal asymptote as $y = 0$ (confusing with degree-1-over-degree-2 cases) are applying the wrong rule. When $\deg(\text{numerator}) = \deg(\text{denominator})$, the horizontal asymptote is the ratio of leading coefficients, not $y = 0$.

Question 5 Hard [6 marks]

The graph of $y = \sqrt{x}$ is transformed to give a new graph with equation $y = g(x)$. The transformations applied, in order, are:

1. Shift left 1 unit.
2. Stretch vertically by factor 3.
3. Shift down 2 units.

(a) Find $g(x)$.

(b) State the domain and range of g.

(c) Find $g^{-1}(x)$ and state its domain.

MISTAKE ANALYSIS

(a) Step 1: $y = \sqrt{x+1}$. Step 2: $y = 3\sqrt{x+1}$. Step 3: $y = 3\sqrt{x+1} - 2$. So $g(x) = 3\sqrt{x+1} - 2$. (b) Domain: $x \geq -1$ (need $x + 1 \geq 0$). Range: $y \geq -2$ (since $3\sqrt{x+1} \geq 0$). (c) $y = 3\sqrt{x+1} - 2 \Rightarrow \frac{y+2}{3} = \sqrt{x+1} \Rightarrow x = \left(\frac{y+2}{3}\right)^2 - 1$. $g^{-1}(x) = \left(\frac{x+2}{3}\right)^2 - 1$. Domain: $x \geq -2$ (= range of g). Students who apply the transformations in the wrong order get a different function. The transformations must be applied to x (inside the function) or to y (outside) in the stated sequence. Shift left means replace x with $x + 1$ (inside); stretch vertically means multiply the whole expression by 3.

Question 6 Hard [6 marks]

Find the equations of all asymptotes of $f(x) = \dfrac{x^2 + 2x - 1}{x - 1}$.

MISTAKE ANALYSIS

Vertical asymptote: $x = 1$ *(denominator zero, numerator* $\neq 0$ *at* $x = 1$*:* $1 + 2 - 1 = 2 \neq 0$*). Since* $\deg(num) = \deg(denom) + 1$*, there is an oblique (slant) asymptote. Polynomial long division:* $x^2 + 2x - 1 = (x-1)(x+3) + 2$*. So* $f(x) = x + 3 + \dfrac{2}{x-1}$*. As* $x \to \pm\infty$*:* $\frac{2}{x-1} \to 0$*, so the oblique asymptote is* $y = x + 3$*. Students who state the asymptote as* $y = x$ *(using only the leading term) are incomplete. Long division gives the exact oblique asymptote* $y = x + 3$*; partial use of the quotient is not sufficient.*

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

(a) Right 2, up 3: $(1, 3) \to (3, 6)$; asymptote $y = 0 \to y = 3$ **A1**

(b) Reflect x-axis, $\times 2$ vertically: $(1, 3) \to (1, -6)$; asymptote unchanged **A1**

(c) Compress horizontally $\times 3$: $(1, 3) \to (1/3, 3)$; asymptote unchanged **A1**

Solution – Question 2

(a) VA: $x = -3$; HA: $y = 2$ (ratio of leading coefficients) **A1**

(b) x-int: $(1/2,\ 0)$; y-int: $(0,\ -1/3)$ **A1**

(c) Two branches about $x = -3$, approaching $y = 2$; graph never crosses HA **R1**

Solution – Question 3

(a) $\lvert f(x)\rvert$: reflect portion $[-2,2]$ upward; touches x-axis at ± 2, local max $(0,4)$	Range: $[0,\infty)$	**M1**
(b) $f(\lvert x\rvert)$: even function, symmetric about y-axis; touches x-axis at ± 2, min $(0,-4)$	Range: $[-4,\infty)$	**M1**

Solution – Question 4

VA: $x=\pm 2$; HA: $y=1$; x-int: ± 1; y-int: $1/4$		**M1**
Three branches; middle branch below $y = 1$ through $(\pm 1, 0)$ and $(0, 1/4)$		**A1**

Solution – Question 5

(a) $\sqrt{x} \to \sqrt{x+1} \to 3\sqrt{x+1} \to 3\sqrt{x+1}-2$	$g(x) = 3\sqrt{x+1}-2$	**M1**
(b) Domain: $x \geq -1$; Range: $y \geq -2$		**A1**
(c) $g^{-1}(x) = \left(\frac{x+2}{3}\right)^2 - 1$; domain $x \geq -2$		**M1**

Solution – Question 6

VA: $x=1$; long division: $x^2+2x-1 = (x-1)(x+3)+2$	$f(x) = x+3+\dfrac{2}{x-1}$	**M1**
Oblique asymptote	$y = x+3$	**A1**
