

# Exponential, Logarithmic & Modulus Equations

*Mistake Analysis – Set III*

<b>Course</b>	IB Mathematics: Analysis & Approaches HL
<b>Topic</b>	Topic 2 – Functions
<b>Level</b>	Medium → Hard
<b>Questions</b>	6
<b>Total marks</b>	34
<b>Instructions</b>	No calculator unless stated. Show all working. <b>M1</b> = method mark. <b>A1</b> = accuracy mark. <b>R1</b> = reasoning mark.

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## BEFORE YOU BEGIN

**Exponential equations:** if both sides can be written as powers of the same base, equate exponents. Otherwise take logarithms of both sides.

**Quadratic-in-disguise:** equations of the form  $a^{2x} + ba^x + c = 0$  become quadratic in  $u = a^x$ .

**Logarithm laws:**  $\log(AB) = \log A + \log B$ ;  $\log(A/B) = \log A - \log B$ ;  $\log(A^n) = n \log A$ . The domain of  $\log A$  requires  $A > 0$ : always check solutions for validity.

**Modulus equations:**  $|f(x)| = g(x)$  gives  $f(x) = g(x)$  or  $f(x) = -g(x)$ . Check all solutions.

**Modulus inequalities:**  $|f(x)| < k$  gives  $-k < f(x) < k$ .

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## Question 1

Medium

[5 marks]

Solve  $2^{2x} - 5 \cdot 2^x + 4 = 0$ .

### MISTAKE ANALYSIS

*Let  $u = 2^x$ :  $u^2 - 5u + 4 = 0$ , so  $(u - 1)(u - 4) = 0$ .  $u = 1 \Rightarrow 2^x = 1 \Rightarrow x = 0$ .  $u = 4 \Rightarrow 2^x = 4 \Rightarrow x = 2$ . Students who attempt to take  $\log_2$  of  $2^{2x} - 5 \cdot 2^x + 4 = 0$  directly cannot separate the terms – logarithms do not distribute over sums. The substitution  $u = 2^x$  is essential: it converts the equation into a standard quadratic. Note that  $u = 2^x > 0$  always, so both solutions  $u = 1$  and  $u = 4$  are valid.*

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**Question 2**

Medium

[5 marks]

Solve  $\log_2 x + \log_2(x - 2) = 3$ .**MISTAKE ANALYSIS**

*Combine:  $\log_2[x(x - 2)] = 3$ , so  $x(x - 2) = 8$ .  $x^2 - 2x - 8 = 0$ , giving  $(x - 4)(x + 2) = 0$ .  $x = 4$  or  $x = -2$ . Check domain:  $\log_2 x$  requires  $x > 0$  and  $\log_2(x - 2)$  requires  $x > 2$ .  $x = -2$  is rejected.  $x = 4$ :  $\log_2 4 + \log_2 2 = 2 + 1 = 3$ . ✓ Students who accept  $x = -2$  without checking fail to recognise that the original equation is undefined at  $x = -2$  (logarithm of a negative number). Always substitute every solution back into the original equation to verify.*

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**Question 3**

Medium

[5 marks]

Solve  $|2x - 3| = x + 1$ .**MISTAKE ANALYSIS**

*Case 1:  $2x - 3 = x + 1 \Rightarrow x = 4$ . Check:  $|5| = 5 = 4 + 1$ . ✓ Case 2:  $-(2x - 3) = x + 1 \Rightarrow -2x + 3 = x + 1 \Rightarrow 2 = 3x \Rightarrow x = \frac{2}{3}$ . Check:  $|4/3 - 3| = |-5/3| = 5/3 = 2/3 + 1$ . ✓ Both solutions are valid. Students who square both sides:  $(2x - 3)^2 = (x + 1)^2 \Rightarrow 4x^2 - 12x + 9 = x^2 + 2x + 1 \Rightarrow 3x^2 - 14x + 8 = 0 \Rightarrow (3x - 2)(x - 4) = 0$ . This also gives  $x = 4$  and  $x = 2/3$  - squaring works here, but introduces a risk of extraneous solutions in other problems. The case-by-case method is safer.*

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**Question 4**

Medium-Hard

[6 marks]

Solve  $e^{2x} - 3e^x - 4 = 0$ .**MISTAKE ANALYSIS**

*Let  $u = e^x$  (so  $e^{2x} = u^2$ ):  $u^2 - 3u - 4 = 0$ , giving  $(u - 4)(u + 1) = 0$ .  $u = 4 \Rightarrow e^x = 4 \Rightarrow x = \ln 4$ .  $u = -1$ : rejected, since  $e^x > 0$  for all real  $x$ . Students who take  $x = \ln(-1)$  do not recognise that this is undefined in the reals.  $e^x > 0$  always: any solution giving  $e^x \leq 0$  must be rejected immediately. Always state the reason for rejection: “ $e^x > 0$  for all real  $x$ , so  $u = -1$  is impossible.” This reasoning earns the R1 mark on the IB mark scheme.*

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**Question 5**

Hard

[6 marks]

Solve  $\log(x + 2) - \log(x - 1) = 1$ , where  $\log$  denotes  $\log_{10}$ .

**MISTAKE ANALYSIS**

$\log\left(\frac{x+2}{x-1}\right) = 1$ , so  $\frac{x+2}{x-1} = 10$ .  $x+2 = 10(x-1) = 10x-10$ , giving  $9x = 12$ , so  $x = \frac{4}{3}$ . Check:  $x+2 = 10/3 > 0 \checkmark$ ;  $x-1 = 1/3 > 0 \checkmark$ . Valid.  $\log(10/3) - \log(1/3) = \log(10) = 1$ .  $\checkmark$  Students who write  $\log(x+2) - \log(x-1) = \log[(x+2) - (x-1)] = \log(3) = 1$  apply the log law incorrectly.  $\log A - \log B = \log(A/B)$ , not  $\log(A-B)$ . The subtraction law for logarithms involves division of the arguments, not subtraction.

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**Question 6**

Hard

[7 marks]

Solve the inequality  $|x^2 - 4| < 3$ .

**MISTAKE ANALYSIS**

$|x^2 - 4| < 3 \Rightarrow -3 < x^2 - 4 < 3$ . Left:  $x^2 > 1 \Rightarrow |x| > 1 \Rightarrow x < -1$  or  $x > 1$ . Right:  $x^2 < 7 \Rightarrow |x| < \sqrt{7} \Rightarrow -\sqrt{7} < x < \sqrt{7}$ . Intersection:  $x \in (-\sqrt{7}, -1) \cup (1, \sqrt{7})$ . Students who only solve  $x^2 - 4 < 3$  (right inequality, giving  $x^2 < 7$ ) and ignore  $x^2 - 4 > -3$  (left inequality, giving  $x^2 > 1$ ) find only half the solution. The modulus inequality  $|A| < k$  always generates two conditions:  $A < k$  **and**  $A > -k$ . Both must be satisfied simultaneously.

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## WORKED SOLUTIONS – SET III – EXPONENTIAL, LOGARITHMIC & MODULUS EQUATIONS

M1 = method mark. A1 = accuracy mark. R1 = reasoning mark.

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### Solution – Question 1

$$\text{Let } u = 2^x: u^2 - u = 1 \text{ or } u = 4 \quad \text{M1}$$

$$5u + 4 = (u - 1)(u - 4) = 0$$

$$2^x = 1 \Rightarrow x = 0; \quad \text{A1}$$

$$2^x = 4 \Rightarrow x = 2$$

**Final answer:**  $x = 0$  or  $x = 2$

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### Solution – Question 2

$$\log_2[x(x - 2)] = x^2 - 2x - 8 = (x - 4)(x + 2) = 0 \quad \text{M1}$$

$$3 \Rightarrow x(x - 2) = 8$$

$$x = 4 \text{ or } x = -2; \quad x = 4 \quad \text{R1}$$

$$\text{reject } x = -2$$

$$(\text{domain: } x > 2)$$

$$\text{Verify: } \log_2 4 + \log_2 2 = 2 + 1 = 3 \quad \text{A1}$$

$$\log_2 2 = 1$$

**Final answer:**  $x = 4$

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### Solution – Question 3

$$\text{Case 1: } 2x - 3 = x + 1 \Rightarrow x = 4 \quad \text{M1}$$

$$x + 1$$

$$\text{Case 2: } -(2x - 3) = x + 1 \Rightarrow x = \frac{2}{3} \quad \text{M1}$$

$$3) = x + 1$$

**Final answer:**  $x = 4$  or  $x = \frac{2}{3}$

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### Solution – Question 4

$$\text{Let } u = e^x: u^2 - 3u - 4 = 0 \text{ or } u = -1 \quad \text{M1}$$

$$3u - 4 = (u - 4)(u + 1) = 0$$

$$4)(u + 1) = 0$$

$$u = -1 \text{ rejected} \quad \text{R1}$$

$$(e^x > 0); e^x = 4$$

$$4 \Rightarrow x = \ln 4$$

**Final answer:**  $x = \ln 4$

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**Solution – Question 5**

$$\log\left(\frac{x+2}{x-1}\right) = 1 \Rightarrow 9x = 12 \Rightarrow x = \frac{4}{3}$$

**M1**

$$\frac{x+2}{x-1} = 10$$

$$\text{Check: } x + 2 = \checkmark$$

**R1**

$$10/3 > 0, x -$$

$$1 = 1/3 >$$

$$0, \log(10/3) -$$

$$\log(1/3) = 1$$

**Final answer:**  $x = \frac{4}{3}$

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**Solution – Question 6**

$$-3 < x^2 - 4 < 3:$$

**M1**

$$\text{left gives } x^2 > 1;$$

$$\text{right gives } x^2 <$$

7

$$|x| > 1 \text{ and } |x| < \sqrt{7} \Rightarrow x \in (-\sqrt{7}, -1) \cup (1, \sqrt{7})$$

**A1**

**Final answer:**  $x \in (-\sqrt{7}, -1) \cup (1, \sqrt{7})$

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