Composite & Inverse Functions

Mistake Analysis – Set I

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 2 – Functions
Level	Easy → Medium
Questions	6
Total marks	31
Instructions	No calculator unless stated. Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

BEFORE YOU BEGIN

The **composite function** $f \circ g$ means $f(g(x))$: apply g first, then f. Note: $f \circ g \neq g \circ f$ in general.

The **inverse function** f^{-1} satisfies $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. f^{-1} exists if and only if f is **one-to-one** (injective) on its domain.

To find f^{-1}: write $y = f(x)$, rearrange for x in terms of y, then swap labels.

The **domain** of f^{-1} equals the **range** of f, and vice versa.

Question 1 Easy [4 marks]

Let $f(x) = 2x + 3$ and $g(x) = x^2 - 1$.

(a) Find $(f \circ g)(x)$.

(b) Find $(g \circ f)(x)$.

(c) Explain why $f \circ g \neq g \circ f$.

MISTAKE ANALYSIS

(a) $(f \circ g)(x) = f(g(x)) = f(x^2 - 1) = 2(x^2 - 1) + 3 = 2x^2 + 1$. *(b)* $(g \circ f)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2 - 1 = 4x^2 + 12x + 8$. *(c)* $2x^2 + 1 \neq 4x^2 + 12x + 8$ *in general (e.g. at* $x = 1$*:* $3 \neq 24$*). The most common error: computing* $f \circ g$ *as* $f(x) \cdot g(x)$ *(multiplication) rather than* $f(g(x))$ *(substitution). The circle notation* $\circ$ *means "apply* g *first, then apply* f *to the result" – not a product.*

Question 2 Easy [4 marks]

Let $f(x) = 3x - 5$.

(a) Find $f^{-1}(x)$.

(b) Find $f^{-1}(7)$.

(c) Verify that $f(f^{-1}(7)) = 7$.

MISTAKE ANALYSIS

(a) Set $y = 3x - 5$*:* $x = \dfrac{y+5}{3}$*. So* $f^{-1}(x) = \dfrac{x+5}{3}$*. (b)* $f^{-1}(7) = \dfrac{12}{3} = 4$*. (c)* $f(4) = 12 - 5 = 7$*.* ✓ *Students who write* $f^{-1}(x) = \dfrac{1}{3x-5}$ *(the reciprocal of* f*) are confusing* f^{-1} *(inverse function) with* $[f(x)]^{-1}$ *(reciprocal).* f^{-1} *means the function that undoes* f*; it has nothing to do with division.*

Question 3 Easy–Medium [5 marks]

Let $f(x) = x^2$ with domain $x \geq 0$.

(a) Find $f^{-1}(x)$ and state its domain and range.

(b) Explain why the domain restriction $x \geq 0$ is necessary.

MISTAKE ANALYSIS

(a) $f^{-1}(x) = \sqrt{x}$*. Domain:* $x \geq 0$*. Range:* $y \geq 0$*. (b) Without the restriction,* $f(x) = x^2$ *is not one-to-one:* $f(2) = f(-2) = 4$*, so we cannot determine a unique* x *from* $f(x)$*. The inverse exists only when* f *is one-to-one. Students who write* $f^{-1}(x) = \pm\sqrt{x}$ *are not defining a function (it maps one input to two outputs). The inverse must be a function: choose the branch consistent with the restricted domain – here* $f^{-1}(x) = \sqrt{x} \geq 0$*.*

Question 4 Medium [5 marks]

Let $f(x) = \dfrac{2x+1}{x-3}$, $x \neq 3$.

(a) Find $f^{-1}(x)$.

(b) State the domain and range of f^{-1}.

(c) Find $f^{-1}(5)$ and verify using f.

MISTAKE ANALYSIS

(a) $y = \dfrac{2x+1}{x-3}$: $y(x-3) = 2x+1$, so $xy - 3y = 2x+1$, giving $x(y-2) = 3y+1$, hence $f^{-1}(x) = \dfrac{3x+1}{x-2}$. (b) Domain of f^{-1}: $x \neq 2$ (since f^{-1} is undefined at $x = 2$). This equals the range of f, which is all reals except 2 (as $f(x) = 2$ has no solution: $2x+1 = 2x-6$ gives $1 = -6$, impossible). Range of f^{-1}: $y \neq 3$ = domain of f. (c) $f^{-1}(5) = \dfrac{16}{3}$. Verify: $f\left(\dfrac{16}{3}\right) = \dfrac{32/3+1}{16/3-3} = \dfrac{35/3}{7/3} = 5$. ✓ Students who state domain of f^{-1} as $x \neq 3$ (copying f's restriction) apply the wrong domain. The domain of f^{-1} is the range of f, not the domain of f.

Question 5 Medium [6 marks]

Let $f(x) = \sqrt{x-2}$ (domain $x \geq 2$) and $g(x) = x^2 + 2$ (domain $\mathbb{R}$).

(a) Find $(f \circ g)(x)$ and state its domain.

(b) Find $(g \circ f)(x)$ and state its domain.

(c) Comment on the relationship between $(g \circ f)(x)$ and x.

MISTAKE ANALYSIS

(a) $(f \circ g)(x) = f(x^2+2) = \sqrt{x^2+2-2} = \sqrt{x^2} = |x|$. Domain: all $x \in \mathbb{R}$ (since $g(x) = x^2+2 \geq 2$ for all x, so f is always defined). (b) $(g \circ f)(x) = g(\sqrt{x-2}) = (\sqrt{x-2})^2 + 2 = x - 2 + 2 = x$. Domain: $x \geq 2$ (the domain of f). (c) On its domain $x \geq 2$: $(g \circ f)(x) = x$ – g is the inverse of f (on the appropriate domain). Students who simplify $\sqrt{x^2} = x$ (dropping the absolute value in part (a)) are wrong for negative x. $\sqrt{x^2} = |x|$, not x. The square root always returns a non-negative value.

Question 6 Medium [7 marks]

The function $h(x) = x^2 - 4x + 7$ is defined for $x \in \mathbb{R}$.

(a) Write $h(x)$ in the form $(x-a)^2 + b$.

(b) State the minimum value of h and the value of x at which it occurs.

(c) Find a domain restriction for h such that h^{-1} exists, and find $h^{-1}(x)$ on that domain.

MISTAKE ANALYSIS

(a) $h(x) = (x-2)^2 + 3$. (b) Minimum value: 3, occurring at $x = 2$. (c) Restrict to $x \geq 2$ (or $x \leq 2$). On $x \geq 2$: $y = (x-2)^2+3 \Rightarrow (x-2)^2 = y-3 \Rightarrow x-2 = \sqrt{y-3}$ (positive root since $x \geq 2$). $h^{-1}(x) = 2+\sqrt{x-3}$, domain $x \geq 3$ (= range of h on $x \geq 2$). Students who choose the domain restriction correctly but then write $x - 2 = \pm\sqrt{y-3}$ take both branches, producing a relation, not a function. The domain restriction determines which square root to take: $x \geq 2$ means $x - 2 \geq 0$, so take the positive root only.

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

(a) $(f \circ g)(x) = f(x^2 - 1) = 2(x^2 - 1) + 3$	$2x^2 + 1$	**M1**
(b) $(g \circ f)(x) = g(2x+3) = (2x+3)^2 - 1$	$4x^2 + 12x + 8$	**M1**
(c) $2x^2 + 1 \neq 4x^2 + 12x + 8$: composition is not commutative	✓	**R1**

Solution – Question 2

(a) $y = 3x - 5 \Rightarrow x = \frac{y+5}{3}$	$f^{-1}(x) = \dfrac{x+5}{3}$	**M1**
(b) $f^{-1}(7) = 12/3$	4	**A1**
(c) $f(4) = 12 - 5 = 7$	✓	**R1**

Solution – Question 3

(a) $y = x^2 \Rightarrow x = \sqrt{y}$	$f^{-1}(x) = \sqrt{x}$, domain $x \geq 0$, range $y \geq 0$	**M1**
(b) Without restriction: $f(2) = f(-2) = 4$; not one-to-one	Inverse would not be a function	**R1**

Solution – Question 4

(a) $x(y-2) = 3y+1$ $f^{-1}(x) = \dfrac{3x+1}{x-2}$ **M1**

(b) Domain: $x \neq 2$; Range: $y \neq 3$ **A1**

(c) $f^{-1}(5) = 16/3$; $f(16/3) = 5$ ✓ **A1**

Solution – Question 5

(a) $(f \circ g)(x) = \sqrt{x^2} = |x|$; domain $\mathbb{R}$ **M1**

(b) $(g \circ f)(x) = (x-2)+2 = x$; domain $x \geq 2$ **M1**

(c) $g \circ f =$ identity on $x \geq 2$: g inverts f ✓ **R1**

Solution – Question 6

(a) $(x-2)^2 + 3$ **M1**

(b) Minimum $= 3$ at $x = 2$ **A1**

(c) Domain $x \geq 2$: $y = (x-2)^2 + 3 \Rightarrow x = 2 + \sqrt{y-3}$ $h^{-1}(x) = 2 + \sqrt{x-3}$, domain $x \geq 3$ **M1**
