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Optimisation & Related Rates

Mistake Analysis – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 5 – Calculus: Differentiation
Level	Medium → Hard
Questions	6
Marks	36 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Optimisation method: (1) Express the quantity to optimise as a function of one variable. (2) Differentiate. (3) Set derivative to zero and solve. (4) Verify it is a maximum or minimum using the second derivative or a sign change.

Related rates method: (1) Write an equation relating the quantities. (2) Differentiate both sides with respect to time t using the chain rule. (3) Substitute the known values and solve for the unknown rate.

Always state units in related rates problems.

Question 1 — Medium — [5 marks]

A rectangular garden is to be enclosed with 40 m of fencing. Find the dimensions that maximise the area, and state the maximum area.

MISTAKE ANALYSIS

Let width $= x$, length $= y$. Constraint: $2x + 2y = 40$, so $y = 20 - x$. Objective: $A = xy = x(20 - x) = 20x - x^2$. $\frac{dA}{dx} = 20 - 2x = 0 \Rightarrow x = 10$, $y = 10$. Maximum area $= 100\,m^2$. Verify: $\frac{d^2A}{dx^2} = -2 < 0$ (maximum). ✓ Students who forget to verify that the critical point is a maximum (not a minimum) lose the R1 mark. In a closed interval context, also check endpoints: here $x = 0$ and $x = 20$ give $A = 0$, confirming the interior critical point is the maximum.

Question 2 — Medium — [6 marks]

A closed cylindrical can has volume $250\pi\,\text{cm}^3$. Find the radius and height that minimise the total surface area.

MISTAKE ANALYSIS

$V = \pi r^2 h = 250\pi \Rightarrow h = \dfrac{250}{r^2}$. $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \dfrac{500\pi}{r}$. $\frac{dS}{dr} = 4\pi r - \dfrac{500\pi}{r^2} = 0 \Rightarrow r^3 = 125 \Rightarrow r = 5$ cm. $h = \dfrac{250}{25} = 10$ cm. Minimum surface area $= 150\pi$ cm^2. $\frac{d^2S}{dr^2} = 4\pi + \dfrac{1000\pi}{r^3} > 0$ (minimum). ✓ Students who forget to include both circular ends in the surface area formula write $S = \pi r^2 + 2\pi rh$ (one end missing). A closed cylinder has two circular ends: $S = 2\pi r^2 + 2\pi rh$.

Question 3 Medium [5 marks]

A company sells x units of a product. The profit function is $P(x) = -x^2 + 80x - 1200$ for $x \geq 0$. Find the number of units that maximises profit, and state the maximum profit.

MISTAKE ANALYSIS

$P'(x) = -2x + 80 = 0 \Rightarrow x = 40$. $P(40) = -1600 + 3200 - 1200 = 400$. $P''(40) = -2 < 0$ (maximum). ✓ Maximum profit is \$400 at $x = 40$ units. Students who set $P(x) = 0$ (finding break-even) rather than $P'(x) = 0$ (finding maximum) solve the wrong equation. The maximum profit occurs where the derivative is zero, not where profit is zero.

Question 4 Medium–Hard [6 marks]

A rectangle is inscribed in a circle of radius R, with its vertices on the circle. Show that the rectangle of maximum area is a square, and find the maximum area.

MISTAKE ANALYSIS

Place the circle at the origin: $x^2 + y^2 = R^2$. Let one vertex be at (x, y); the rectangle has sides $2x$ and $2y$. Area $A = 4xy$. Constraint: $y = \sqrt{R^2 - x^2}$. $A = 4x\sqrt{R^2 - x^2}$. $\frac{dA}{dx} = 4\left(\sqrt{R^2 - x^2} - \frac{x^2}{\sqrt{R^2 - x^2}}\right) = 0$. $R^2 - x^2 = x^2 \Rightarrow x = \frac{R}{\sqrt{2}}$, so $y = \frac{R}{\sqrt{2}}$: a square. ✓ Maximum area $= 4 \cdot \frac{R}{\sqrt{2}} \cdot \frac{R}{\sqrt{2}} = 2R^2$. Students who parametrise with $x = R\cos\theta$, $y = R\sin\theta$ and then maximise $A = 4R^2 \sin\theta\cos\theta = 2R^2 \sin 2\theta$: this is elegant and correct – maximum at $\theta = \pi/4$ giving $A = 2R^2$.

Question 5 Hard [6 marks]

Air is being pumped into a spherical balloon at a rate of $100\,\text{cm}^3\text{s}^{-1}$. Find the rate at which the radius is increasing when the radius is $5\,\text{cm}$.

MISTAKE ANALYSIS

$V = \frac{4}{3}\pi r^3$. Differentiate with respect to t: $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. When $r = 5$: $100 = 4\pi(25)\frac{dr}{dt}$, so $\frac{dr}{dt} = \frac{100}{100\pi} = \frac{1}{\pi}\,cm\,s^{-1}$. The chain rule gives $\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt}$. Students who forget to differentiate with respect to t (writing $V = \frac{4}{3}\pi r^3$ and $\frac{dV}{dt} = 4\pi r^2$ without the factor $\frac{dr}{dt}$) omit the chain rule. The rate of change of r is what is being found; it cannot be dropped.

Question 6 Hard [8 marks]

A ladder of length $10\,\text{m}$ leans against a vertical wall. The base of the ladder slides away from the wall at $0.5\,\text{m}\,\text{s}^{-1}$.

(a) Find the rate at which the top of the ladder is sliding down the wall when the base is $6\,\text{m}$ from the wall.

(b) Find the rate at which the area of the triangle formed by the ladder, wall, and ground is changing at this instant.

MISTAKE ANALYSIS

$x^2 + y^2 = 100$. At $x = 6$: $y = 8$. (a) Differentiate: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$, so $\frac{dy}{dt} = -\frac{x}{y}\cdot\frac{dx}{dt} = -\frac{6}{8}\cdot 0.5 = -\frac{3}{8}\,m\,s^{-1}$. The negative sign means the top is sliding down. (b) $A = \frac{1}{2}xy$. $\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) = \frac{1}{2}\left(6\cdot\left(-\frac{3}{8}\right) + 8\cdot 0.5\right) = \frac{1}{2}(-\frac{9}{4}+4) = \frac{7}{8}\,m^2s^{-1}$. Students who ignore the sign in part (a) and state $\frac{3}{8}\,m\,s^{-1}$ upward are wrong about direction. The sign of $\frac{dy}{dt}$ carries physical meaning: negative means decreasing (sliding down).

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

$y = 20 - x;\ A = x(20 - x);\ A' = 20 - 2x = 0$	$x = 10,\ y = 10$	**M1**
$A'' = -2 < 0$: maximum	Max area $= 100\,\mathrm{m}^2$	**R1**

Solution – Question 2

$h = 250/r^2;\ S = 2\pi r^2 + 500\pi/r;\ S' = 4\pi r - 500\pi/r^2 = 0$	$r^3 = 125 \Rightarrow r = 5$	**M1**
$h = 10;\ S'' = 4\pi + 1000\pi/r^3 > 0$: minimum	Min $S = 150\pi\,\mathrm{cm}^2$	**A1**

Solution – Question 3

$P'(x) = -2x + 80 = 0$	$x = 40$	**M1**
$P(40) = 400;\ P'' = -2 < 0$: maximum	Max profit \$400	**A1**

Solution – Question 4

$A = 4xy = 4x\sqrt{R^2 - x^2};\ A' = 0 \Rightarrow R^2 - 2x^2 = 0$	$x = y = R/\sqrt{2}$: square ✓	**M1**
Max area	$2R^2$	**A1**

Solution – Question 5

$\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt}$; $\dfrac{dr}{dt} = \dfrac{1}{\pi}\,\mathrm{cm\,s^{-1}}$ **M1**
at $r = 5$: $100 = 100\pi \dfrac{dr}{dt}$

Solution – Question 6

(a) $2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} = 0$; at $x = 6$, $y = 8$, $\dfrac{dx}{dt} = 0.5$ $\dfrac{dy}{dt} = -\dfrac{3}{8}\,\mathrm{m\,s^{-1}}$ (downward) **M1**

(b) $\dfrac{dA}{dt} = \frac{1}{2}\left(x\dfrac{dy}{dt} + y\dfrac{dx}{dt}\right) = \frac{1}{2}(-\frac{9}{4} + 4)$ $= \dfrac{7}{8}\,\mathrm{m^2 s^{-1}}$ **M1**
