L'Hôpital's Rule, Inverse Trig & Higher-Order Derivatives

Mistake Analysis – Set III

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 5 – Calculus: Differentiation
Level	Medium → Hard
Questions	6
Marks	34 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

L'Hôpital's Rule: If $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$ (provided this limit exists). State the indeterminate form before applying the rule.

Inverse trig derivatives: $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$; $\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$; $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$.

Higher-order derivatives: $f''(x)$ is the derivative of $f'(x)$; $f^{(n)}(x)$ is the nth derivative.

Question 1 — Medium — [5 marks]

Evaluate the following limits using L'Hôpital's Rule where applicable.

(a) $\lim_{x\to 0}\frac{\sin x}{x}$

(b) $\lim_{x\to 0}\frac{e^x - 1}{x}$

(c) $\lim_{x\to\infty} xe^{-x}$

MISTAKE ANALYSIS

(a) Form $\frac{0}{0}$: L'Hôpital gives $\lim_{x\to 0}\frac{\cos x}{1} = 1$. (b) Form $\frac{0}{0}$: $\lim_{x\to 0}\frac{e^x}{1} = 1$. (c) Rewrite as $\frac{x}{e^x}$, form $\frac{\infty}{\infty}$: $\lim_{x\to\infty}\frac{1}{e^x} = 0$. Students who apply L'Hôpital without first verifying the indeterminate form earn no M1. The rule only applies to $\frac{0}{0}$ or $\frac{\infty}{\infty}$. For (c): xe^{-x} is not immediately in fraction form; rewrite as $\frac{x}{e^x}$ first. Other forms ($0\cdot\infty$, $\infty-\infty$, 0^0) require algebraic manipulation before the rule applies.

Question 2

Medium

[5 marks]

Evaluate $\lim_{x \to 0} \frac{e^x - 1 - x}{x^2}$.

MISTAKE ANALYSIS

Form $\frac{0}{0}$: differentiate numerator and denominator. $\lim_{x\to 0} \frac{e^x-1}{2x}$: still $\frac{0}{0}$. Apply again: $\lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$. L'Hôpital can be applied repeatedly, provided the indeterminate form persists. Students who stop after one application and conclude the limit is $\frac{0}{0}$ (undefined) have not completed the process. Re-check the form after each application. This limit is also obtainable from the Maclaurin series $e^x = 1 + x + \frac{x^2}{2} + \cdots$, giving $\frac{x^2/2+\cdots}{x^2} \to \frac{1}{2}$.

Question 3

Medium

[5 marks]

Differentiate each function.

(a) $f(x) = \arcsin(3x)$

(b) $g(x) = \arctan(x^2)$

(c) $h(x) = x \arctan x$

MISTAKE ANALYSIS

(a) Chain rule: $f'(x) = \dfrac{1}{\sqrt{1-(3x)^2}} \cdot 3 = \dfrac{3}{\sqrt{1-9x^2}}$. Domain: $|x| < \frac{1}{3}$. (b) Chain rule: $g'(x) = \dfrac{1}{1+(x^2)^2} \cdot 2x = \dfrac{2x}{1+x^4}$. At $x = 1$: $g'(1) = 1$. (c) Product rule: $h'(x) = \arctan x + \dfrac{x}{1+x^2}$. For (a): students who write $\dfrac{1}{\sqrt{1-9x^2}}$ (omitting the factor 3 from the chain rule) forget the inner derivative. For (b): $(x^2)^2 = x^4$, not x^2. Squaring x^2 gives x^4.

Question 4

Medium–Hard

[6 marks]

Let $y = x^4 - 3x^2 + 2$.

(a) Find y', y'', y''', and $y^{(4)}$.

(b) Find the values of x where $y'' = 0$ and state what this means geometrically.

MISTAKE ANALYSIS

*(a) $y' = 4x^3 - 6x$; $y'' = 12x^2 - 6$; $y''' = 24x$; $y^{(4)} = 24$. (b) $y'' = 0$: $12x^2 - 6 = 0 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$. These are **points of inflection** (where the curve changes concavity), provided $y'''(\pm\frac{1}{\sqrt{2}}) \neq 0$. $y'''(\frac{1}{\sqrt{2}}) = 24/\sqrt{2} \neq 0$. ✓ Students who state that $y'' = 0$ means a minimum or maximum are wrong. A stationary point requires $y' = 0$. $y'' = 0$ is a necessary (not sufficient) condition for a point of inflection.*

Question 5

Hard

[6 marks]

Show that if $y = e^x \sin x$, then $y'' - 2y' + 2y = 0$.

MISTAKE ANALYSIS

$y' = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$. $y'' = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x$. $y'' - 2y' + 2y = 2e^x \cos x - 2e^x(\sin x + \cos x) + 2e^x \sin x = e^x[2\cos x - 2\sin x - 2\cos x + 2\sin x] = 0$. ✓ Students who attempt to verify this by substituting $x = 0$ only (checking one value) have not proved it for all x. A verification at a single point is not a proof. Also: $y'' = e^x(\sin x + \cos x) + e^x(\cos x - \sin x)$; students who differentiate y' without applying the product rule to each term reach the wrong expression.

Question 6

Hard

[7 marks]

(a) Using L'Hôpital's Rule, evaluate $\lim_{x\to 0^+} x \ln x$.

(b) Hence find $\lim_{x\to 0^+} x^x$.

MISTAKE ANALYSIS

(a) $x \ln x = \dfrac{\ln x}{1/x}$, form $\frac{-\infty}{\infty}$. L'Hôpital: $\dfrac{1/x}{-1/x^2} = \dfrac{x^2 \cdot 1}{x} = \dfrac{x \cdot x}{x}$... more carefully: $\dfrac{(1/x)}{(-1/x^2)} = \dfrac{1}{x} \cdot (-x^2) = -x \to 0$ as $x \to 0^+$. So $\lim_{x\to 0^+} x \ln x = 0$. (b) $x^x = e^{x \ln x}$. Since $x \ln x \to 0$: $\lim_{x\to 0^+} x^x = e^0 = 1$. The form 0^0 is indeterminate. Students who state $0^0 = 0$ or $0^0 = 1$ without justification are asserting a result that requires proof. The standard technique is to write $x^x = e^{x \ln x}$ and evaluate the exponent separately using part (a).

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

(a) 0/0: $\frac{\cos x}{1} \to 1$; **M1**
(b) 0/0: $\frac{e^x}{1} \to 1$
(c) $\frac{x}{e^x}$, ∞/∞: $\frac{1}{e^x} \to 0$ **M1**

Solution – Question 2

0/0: $\frac{e^x-1}{2x}$; still 0/0: $\frac{e^x}{2} \to \frac{1}{2}$ $\frac{1}{2}$ **M1**

Solution – Question 3

(a) Chain: $\dfrac{3}{\sqrt{1-9x^2}}$; (b) Chain: $\dfrac{2x}{1+x^4}$ **M1**
(c) Product: $\arctan x + \dfrac{x}{1+x^2}$ **A1**

Solution – Question 4

(a) $y' = 4x^3 - 6x$; $y'' = 12x^2 - 6$; $y''' = 24x$; $y^{(4)} = 24$ **M1**
(b) $y'' = 0 \Rightarrow x = \pm 1/\sqrt{2}$; $y''' \neq 0$: points of inflection ✓ **R1**

Solution – Question 5

$y' = e^x(\sin x + \cos x)$; $y'' = 2e^x \cos x$ **M1**

$y'' - 2y' + 2y = 2e^x \cos x - 2e^x(\sin x + \cos x) + 2e^x \sin x = 0$ ✓ **A1**

Solution – Question 6

(a) $\frac{\ln x}{1/x}$, $-\infty/\infty$: $\frac{1/x}{-1/x^2} = -x \to 0$ **M1**

(b) $x^x = e^{x \ln x} \to e^0 = 1$ ✓ **A1**
