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Separable Differential Equations

Recognition Training – Set I

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 5 – Calculus
Level	Easy → Medium
Questions	6
Total marks	30
Instructions	Show all working. **M1** = method mark (correct process). **A1** = accuracy mark (correct result). **R1** = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

A differential equation is separable if it can be written in the form $\frac{dy}{dx} = f(x)\,g(y)$, allowing the variables to be separated: $\frac{1}{g(y)}\,dy = f(x)\,dx$. Both sides are then integrated independently. The separation step must be written explicitly – do not skip it. When an initial condition is given, substitute it after integrating to find the particular solution. Do not substitute before integrating.

Question 1 — Easy — [4 marks]

Find the general solution of the following differential equation.

$$\frac{dy}{dx} = 2xy$$

WARNING – RECOGNITION TRAP

Separate variables: $\frac{1}{y}\,dy = 2x\,dx$. *Integrate both sides:* $\ln|y| = x^2 + C$. *Exponentiate:* $|y| = e^{x^2+C} = Ae^{x^2}$ *where* $A = e^C > 0$. *Writing* $y = Ae^{x^2}$ *(absorbing sign into* A*, now any non-zero constant). The most common error: students write* $\ln y = x^2 + C$ *and then forget to exponentiate, leaving the answer as* $\ln y = x^2 + C$. *This is implicit form – the IB mark scheme requires* y *isolated unless the question states otherwise.*

Question 2 Easy [4 marks]

Find the general solution of the following differential equation.

$$\frac{dy}{dx} = \frac{x^2}{y}$$

WARNING – RECOGNITION TRAP

Separate: $y\,dy = x^2\,dx$. Integrate: $\frac{y^2}{2} = \frac{x^3}{3} + C$. Students frequently forget to divide y^2 by 2 when integrating the left side, writing $y^2 = \frac{x^3}{3} + C$ instead of $\frac{y^2}{2}$. Also: the constant C here represents $\frac{1}{2}$ times the true constant of integration – this is acceptable. The IB mark scheme accepts any form of the constant provided it is present.

Question 3 Easy–Medium [5 marks]

Find the particular solution of the following differential equation, given that $y = 2$ when $x = 0$.

$$\frac{dy}{dx} = \frac{x}{y^2}$$

WARNING – RECOGNITION TRAP

Separate: $y^2\,dy = x\,dx$. Integrate: $\frac{y^3}{3} = \frac{x^2}{2} + C$. Apply initial condition $y = 2$, $x = 0$: $\frac{8}{3} = 0 + C$, so $C = \frac{8}{3}$. Particular solution: $\frac{y^3}{3} = \frac{x^2}{2} + \frac{8}{3}$, giving $y^3 = \frac{3x^2}{2} + 8$, so $y = \left(\frac{3x^2}{2} + 8\right)^{1/3}$. The error: students substitute the initial condition into the unseparated equation or into one side only. Always integrate first, then apply the initial condition.

Question 4 Medium [5 marks]

Find the general solution of the following differential equation.

$$\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$$

WARNING – RECOGNITION TRAP

Separate: $\dfrac{1}{1+y^2}\,dy = \dfrac{1}{1+x^2}\,dx$. *Integrate both sides:* $\arctan y = \arctan x + C$. *This can be written as* $y = \tan(\arctan x + C)$. *The error: students see* $\dfrac{1+y^2}{1+x^2}$ *and attempt substitution* $u = 1+y^2$ *or* $u = 1+x^2$, *which does not simplify the equation. The structure* $f(y)/f(x)$ *– where both sides have the same function form – is a strong signal that the equation is separable and each side integrates to an arctangent. Recognise the form before reaching for a technique.*

Question 5 Medium [6 marks]

Find the particular solution of the following differential equation, given that $y = 1$ when $x = 1$.

$$\frac{dy}{dx} = \frac{y \ln y}{x}$$

WARNING – RECOGNITION TRAP

Separate: $\dfrac{1}{y\ln y}\,dy = \dfrac{1}{x}\,dx$. *Left side: let* $u = \ln y$, $du = \frac{1}{y}\,dy$, *so* $\int \dfrac{1}{y\ln y}\,dy = \int \dfrac{1}{u}\,du = \ln|\ln y|$. *Right side:* $\ln|x|$. *So* $\ln|\ln y| = \ln|x| + C$. *Apply* $y = 1$, $x = 1$: $\ln|\ln 1| = \ln 1 + C = 0 + C$. *But* $\ln 1 = 0$, *so* $\ln|0|$ *is undefined. Revisit: the initial condition* $y = 1$ *gives* $\ln y = 0$, *which makes* $\dfrac{dy}{dx} = \dfrac{1\cdot 0}{1} = 0$. *The point* $(1,1)$ *is a constant solution. The general solution* $\ln|\ln y| = \ln x + C$ *holds for* $y > 1$ *(where* $\ln y > 0$*). This question tests whether students check the domain of the initial condition before applying it. A sophisticated question with no single "clean" particular solution for this initial condition.*

Question 6 Medium [6 marks]

Find the particular solution of the following differential equation, given that $y = 0$ when $x = \dfrac{\pi}{2}$.

$$\frac{dy}{dx} = y\cos x$$

WARNING – RECOGNITION TRAP

Separate: $\frac{1}{y}\,dy = \cos x\,dx$. *Integrate:* $\ln|y| = \sin x + C$. *Exponentiate:* $y = Ae^{\sin x}$ *where* A *is an arbitrary constant. Apply* $y = 0$, $x = \pi/2$: $0 = Ae^{\sin(\pi/2)} = Ae^1 = Ae$. *Since* $e \neq 0$, *we require* $A = 0$, *giving* $y = 0$. *This is the trivial (zero) solution – valid but not always expected. Students who separate and integrate correctly but then find* $A = 0$ *sometimes discard this answer, believing it must be wrong. It is not wrong.* $y = 0$ *is a valid particular solution satisfying both the equation and the initial condition. IB mark schemes accept it.*

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

Separate	$\frac{1}{y}\,dy = 2x\,dx$	**M1**
Integrate both sides	$\ln\lvert y\rvert = x^2 + C$	**M1**
Exponentiate	$\lvert y\rvert = e^{x^2+C} = e^C \cdot e^{x^2}$	**A1**
Write general solution	$y = Ae^{x^2}$ (A an arbitrary non-zero constant)	**A1**

Final answer: $y = Ae^{x^2}$

Solution – Question 2

Separate	$y\,dy = x^2\,dx$	**M1**
Integrate both sides	$\frac{y^2}{2} = \frac{x^3}{3} + C$	**M1**
Write general solution	$y^2 = \frac{2x^3}{3} + A$ ($A = 2C$, arbitrary)	**A1**

Final answer: $y^2 = \frac{2x^3}{3} + A$

Solution – Question 3

Separate	$y^2\,dy = x\,dx$	**M1**
Integrate	$\frac{y^3}{3} = \frac{x^2}{2} + C$	**M1**
Apply IC: $y = 2$, $x = 0$	$\frac{8}{3} = 0 + C \Rightarrow C = \frac{8}{3}$	**A1**
Particular solution	$\frac{y^3}{3} = \frac{x^2}{2} + \frac{8}{3} \Rightarrow y^3 = \frac{3x^2}{2} + 8$	**A1**
Solve for y	$y = \left(\frac{3x^2}{2} + 8\right)^{1/3}$	**A1**

Final answer: $y = \left(\dfrac{3x^2}{2} + 8\right)^{1/3}$

Solution – Question 4

Separate	$\dfrac{dy}{1+y^2} = \dfrac{dx}{1+x^2}$	**M1**
Integrate both sides	$\arctan y = \arctan x + C$	**M1**
Write general solution	$y = \tan(\arctan x + C)$	**A1**

Final answer: $y = \tan(\arctan x + C)$

Solution – Question 5

Separate	$\dfrac{dy}{y\ln y} = \dfrac{dx}{x}$	**M1**
Left side: let $u = \ln y$, $du = \frac{dy}{y}$	$\displaystyle\int \frac{du}{u} = \ln\|\ln y\|$	**M1**
Integrate both sides	$\ln\|\ln y\| = \ln\|x\| + C$	**A1**
Check IC: $y = 1$, $x = 1$	$\ln y\big\|_{y=1} = \ln 1 = 0 \;\Rightarrow\; \ln\|0\|$ undefined	**R1**
Conclusion	$y = 1$ is a constant (trivial) solution: $\dfrac{dy}{dx} = \dfrac{1 \cdot 0}{1} = 0$✓	**A1**

The general solution $\ln|\ln y| = \ln x + C$ holds for $y > 1$. The initial condition $y(1) = 1$ corresponds to the trivial solution $y \equiv 1$.

Final answer: $y = 1$ (trivial particular solution)

Solution – Question 6

Separate	$\dfrac{dy}{y} = \cos x\,dx$	**M1**
Integrate	$\ln\|y\| = \sin x + C$	**M1**
Exponentiate	$y = Ae^{\sin x}$	**A1**
Apply IC: $y = 0$, $x = \pi/2$	$0 = Ae^1 \;\Rightarrow\; A = 0$	**A1**
Particular solution	$y = 0$	**A1**

Final answer: $y = 0$ (the trivial solution – valid and complete)

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