

Integrating Factor Method

Recognition Training – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 5 – Calculus
Level	Medium → Hard
Questions	6
Total marks	32
Instructions	Show all working. M1 = method mark. A1 = accuracy mark. R1 = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

The integrating factor method applies to linear first-order differential equations of the form $\frac{dy}{dx} + P(x)y = Q(x)$. The integrating factor is $\mu(x) = e^{\int P(x) dx}$. Multiplying both sides by μ converts the left side to $\frac{d}{dx}[\mu y]$, allowing direct integration. Three steps must all be shown explicitly to earn full marks: (1) identifying $P(x)$ and computing μ ; (2) multiplying through and recognising the product rule form; (3) integrating both sides and applying any initial condition. Skipping step (2) – jumping directly from $\mu \cdot \frac{dy}{dx} + \mu P y = \mu Q$ to $\mu y = \int \mu Q dx$ – costs a method mark.

Question 1

Medium

[5 marks]

Find the general solution of the following differential equation.

$$\frac{dy}{dx} + \frac{y}{x} = x^2$$

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$P(x) = \frac{1}{x}$, so $\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$. Multiply through: $x \frac{dy}{dx} + y = x^3$, which is $\frac{d}{dx}[xy] = x^3$. Integrate: $xy = \frac{x^4}{4} + C$. General solution: $y = \frac{x^3}{4} + \frac{C}{x}$. The most common error: computing $\mu = e^{\ln x} = x$ correctly but then writing $\frac{d}{dx}[\mu y] = \mu \frac{dy}{dx} + y$ instead of $\frac{d}{dx}[xy] = x \frac{dy}{dx} + y$ – the product rule requires $\frac{d}{dx}[\mu y] = \mu \frac{dy}{dx} + \mu' y$. Always write the product rule step explicitly.

Question 2

Medium

[5 marks]

Find the general solution of the following differential equation.

$$\frac{dy}{dx} - 2y = e^{3x}$$

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$P(x) = -2$, so $\mu = e^{\int (-2) dx} = e^{-2x}$. Multiply: $e^{-2x} \frac{dy}{dx} - 2e^{-2x}y = e^x$, i.e. $\frac{d}{dx}[e^{-2x}y] = e^x$. Integrate: $e^{-2x}y = e^x + C$. General solution: $y = e^{3x} + Ce^{2x}$. The error: students compute $\mu = e^{-2x}$ correctly then multiply only the left side, writing $e^{-2x} \frac{dy}{dx} - 2e^{-2x}y = e^{3x}$ without multiplying the right side by μ . The right side becomes $e^{-2x} \cdot e^{3x} = e^x$, not e^{3x} . Multiply every term by μ .

Question 3

Medium

[5 marks]

Find the general solution of the following differential equation.

$$\frac{dy}{dx} + y \tan x = \sec x$$

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$P(x) = \tan x$, so $\mu = e^{\int \tan x dx} = e^{-\ln |\cos x|} = \frac{1}{\cos x} = \sec x$. Multiply: $\sec x \frac{dy}{dx} + y \sec x \tan x = \sec^2 x$, i.e. $\frac{d}{dx}[y \sec x] = \sec^2 x$. Integrate: $y \sec x = \tan x + C$. General solution: $y = \sin x + C \cos x$. Students frequently compute $\int \tan x dx$ incorrectly, writing $\sec^2 x$ (the derivative, not the integral). $\int \tan x dx = -\ln |\cos x| = \ln |\sec x|$, so $\mu = e^{\ln |\sec x|} = \sec x$. Know this integral – it appears repeatedly in IB differential equations.

Question 4

Medium–Hard

[6 marks]

Find the particular solution of the following differential equation, given that $y = 2$ when $x = 0$.

$$\frac{dy}{dx} + 2xy = 2x$$

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$P(x) = 2x$, so $\mu = e^{\int 2x dx} = e^{x^2}$. Multiply: $e^{x^2} \frac{dy}{dx} + 2xe^{x^2}y = 2xe^{x^2}$, i.e. $\frac{d}{dx}[ye^{x^2}] = 2xe^{x^2}$. Integrate: $ye^{x^2} = e^{x^2} + C$ (since $\int 2xe^{x^2} dx = e^{x^2}$ by inspection – inner derivative $2x$ is present). Apply $y = 2, x = 0$: $2e^0 = e^0 + C \Rightarrow C = 1$. Particular solution: $ye^{x^2} = e^{x^2} + 1$, so $y = 1 + e^{-x^2}$. The error: students integrate $\int 2xe^{x^2} dx$ by parts instead of recognising it as the reverse chain rule. $\int 2xe^{x^2} dx = e^{x^2} + C$ directly.

Question 5

Hard

[5 marks]

Find the general solution of the following differential equation.

$$x \frac{dy}{dx} + 2y = x^3 \ln x$$

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First divide by x to write in standard form: $\frac{dy}{dx} + \frac{2}{x}y = x^2 \ln x$. $P(x) = \frac{2}{x}$, so $\mu = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$. Multiply: $\frac{d}{dx}[x^2y] = x^4 \ln x$. Integrate right side by parts: $u = \ln x, dv = x^4 dx, \int x^4 \ln x dx = \frac{x^5}{5} \ln x - \frac{x^5}{25} + C$. General solution: $x^2y = \frac{x^5}{5} \ln x - \frac{x^5}{25} + C$, so $y = \frac{x^3}{5} \ln x - \frac{x^3}{25} + \frac{C}{x^2}$. The critical first step students miss: dividing through by x before computing $P(x)$ and μ . If they use $P = 2$ instead of $P = 2/x$, the integrating factor is wrong.

Question 6

Hard

[6 marks]

Find the particular solution of the following differential equation, given that $y = 1$ when $x = 1$.

$$\frac{dy}{dx} + \frac{y}{x} = \frac{\ln x}{x}$$

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$P(x) = \frac{1}{x}$, so $\mu = x$ (same as Q1). Multiply: $\frac{d}{dx}[xy] = \ln x$. Integrate: $xy = \int \ln x dx = x \ln x - x + C$. Apply $y = 1, x = 1$: $1 \cdot 1 = 1 \cdot 0 - 1 + C \Rightarrow C = 2$. Particular solution: $xy = x \ln x - x + 2$, so

$y = \ln x - 1 + \frac{2}{x}$. The integration of $\ln x$ requires integration by parts (or knowledge of the standard result $\int \ln x \, dx = x \ln x - x + C$). Students who do not know this result attempt substitution, which does not simplify $\int \ln x \, dx$. This is a standard result that must be memorised at IB AA HL level.

WORKED SOLUTIONS – SET II – INTEGRATING FACTOR METHOD

M1 = method mark. A1 = accuracy mark. R1 = reasoning mark.

Solution – Question 1

Identify μ $P(x) = \frac{1}{x} \Rightarrow \mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$ **M1**

Multiply through $\frac{d}{dx}[xy] = x^3$ **M1**

Integrate $xy = \frac{x^4}{4} + C$ **A1**

General solution $y = \frac{x^3}{4} + \frac{C}{x}$ **A1**

Final answer: $y = \frac{x^3}{4} + \frac{C}{x}$

Solution – Question 2

Identify μ $P(x) = -2 \Rightarrow \mu = e^{-2x}$ **M1**

Multiply through $\frac{d}{dx}[e^{-2x}y] = e^{-2x} \cdot e^{3x} = e^x$ **M1**

Integrate $e^{-2x}y = e^x + C$ **A1**

General solution $y = e^{3x} + Ce^{2x}$ **A1**

Final answer: $y = e^{3x} + Ce^{2x}$

Solution – Question 3

Identify μ $P(x) = \tan x \Rightarrow \mu = e^{\int \tan x dx} = e^{\ln |\sec x|} = \sec x$ **M1**

Multiply through $\frac{d}{dx}[y \sec x] = \sec^2 x$ **M1**

Integrate $y \sec x = \tan x + C$ **A1**

General solution $y = \sin x + C \cos x$ **A1**

Final answer: $y = \sin x + C \cos x$

Solution – Question 4

Identify μ $P(x) = 2x \Rightarrow \mu = e^{x^2}$ **M1**

Multiply through $\frac{d}{dx}[ye^{x^2}] = 2xe^{x^2}$ **M1**

Integrate (reverse chain rule) $ye^{x^2} = e^{x^2} + C$ **A1**

Apply IC: $y = 2, x = 0$ $2 = 1 + C \Rightarrow C = 1$ **A1**

Particular solution $ye^{x^2} = e^{x^2} + 1 \Rightarrow y = 1 + e^{-x^2}$ **A1**

Final answer: $y = 1 + e^{-x^2}$

Solution – Question 5

Divide by x $\frac{dy}{dx} + \frac{2}{x}y = x^2 \ln x$ **M1**

Identify μ $P(x) = \frac{2}{x} \Rightarrow \mu = e^{2 \ln x} = x^2$ **M1**

Multiply through $\frac{d}{dx}[x^2y] = x^4 \ln x$ **M1**

IBP: $u = \ln x, dv = x^4 dx$ $\int x^4 \ln x dx = \frac{x^5}{5} \ln x - \frac{x^5}{25} + C$ **A1**

General solution $x^2y = \frac{x^5}{5} \ln x - \frac{x^5}{25} + C$ **A1**

Divide by x^2 $y = \frac{x^3}{5} \ln x - \frac{x^3}{25} + \frac{C}{x^2}$ **A1**

Final answer: $y = \frac{x^3}{5} \ln x - \frac{x^3}{25} + \frac{C}{x^2}$

Solution – Question 6

Identify μ $P(x) = \frac{1}{x} \Rightarrow \mu = x$ **M1**

Multiply through $\frac{d}{dx}[xy] = \ln x$ **M1**

Integrate: $\int \ln x dx$ $xy = x \ln x - x + C$ **A1**

Apply IC: $y = 1, x = 1$ $1 = 0 - 1 + C \Rightarrow C = 2$ **A1**

Particular solution $xy = x \ln x - x + 2$ **A1**

Solve for y $y = \ln x - 1 + \frac{2}{x}$ **A1**

Final answer: $y = \ln x - 1 + \frac{2}{x}$

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