Homogeneous & Second Order

Recognition Training – Set III

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 5 – Calculus
Level	Medium → Hard
Questions	6
Total marks	32
Instructions	Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

This set covers two distinct techniques. **Homogeneous equations** have the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$. The substitution $y = vx$ (so $\frac{dy}{dx} = v + x\frac{dv}{dx}$) converts them to separable equations in v and x. **Second order linear equations** with constant coefficients have the form $ay'' + by' + cy = f(x)$. The general solution is $y = y_c + y_p$, where y_c is the complementary function (from the auxiliary equation $am^2 + bm + c = 0$) and y_p is a particular integral found by the method of undetermined coefficients. Both techniques require identifying the equation type before applying any method.

Question 1 — Medium — [5 marks]

Find the general solution of the following differential equation.

$$\frac{dy}{dx} = \frac{y}{x} + 1$$

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Write as $\frac{dy}{dx} = \frac{y}{x} + 1 = f\left(\frac{y}{x}\right)$ – homogeneous. Substitute $y = vx$: $v + x\frac{dv}{dx} = v + 1$, so $x\frac{dv}{dx} = 1$. Separate: $dv = \frac{dx}{x}$. Integrate: $v = \ln|x| + C$. Back-substitute $v = \frac{y}{x}$: $\frac{y}{x} = \ln|x| + C$, so $y = x\ln|x| + Cx$. Students who do not recognise the homogeneous form attempt the integrating factor method, writing $\frac{dy}{dx} - \frac{y}{x} = 1$ with $P = -1/x$, $\mu = 1/x$. This also works and gives the same answer, but the homogeneous substitution is the more natural route. Both methods earn full marks if executed correctly.

Question 2 Medium [5 marks]

Find the general solution of the following differential equation.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$$

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Divide numerator and denominator by x^2: $\frac{dy}{dx} = \frac{1+(y/x)^2}{2(y/x)} = f\left(\frac{y}{x}\right)$ – homogeneous. Substitute $y = vx$: $v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$. So $x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$. Separate: $\frac{2v}{1-v^2}\,dv = \frac{dx}{x}$. Integrate left side: $-\ln|1-v^2| = \ln|x| + C$, so $\ln\frac{1}{|1-v^2|} = \ln|x| + C$, giving $\frac{1}{1-v^2} = Ax$. Back-substitute: $\frac{x^2}{x^2-y^2} = Ax$, so $x^2 - y^2 = \frac{x}{A} = Bx$. General solution: $x^2 - y^2 = Bx$. The error: students fail to recognise the homogeneous structure, or divide incorrectly by x rather than x^2.

Question 3 Medium [5 marks]

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

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Auxiliary equation: $m^2 - 5m + 6 = 0$, factorises as $(m-2)(m-3) = 0$. Roots: $m = 2$, $m = 3$ – two distinct real roots. Complementary function: $y = Ae^{2x} + Be^{3x}$. Since $f(x) = 0$ (homogeneous second order), this is the complete general solution. Students who write $y = Ae^{2x} + Be^{3x} + C$ are incorrect – the constant C is not needed because it would satisfy $y'' - 5y' + 6y = 6C \neq 0$ unless $C = 0$. The general solution of a second order ODE has exactly two arbitrary constants.

Question 4 Medium–Hard [5 marks]

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} + 4y = 0$$

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Auxiliary equation: $m^2 + 4 = 0$, so $m^2 = -4$, giving $m = \pm 2i$. Complex roots $\alpha \pm \beta i$ with $\alpha = 0$, $\beta = 2$. General solution: $y = A\cos 2x + B\sin 2x$. Students who write $y = \mathrm{e}^0(A\cos 2x + B\sin 2x) = A\cos 2x + B\sin 2x$ are correct (the $\mathrm{e}^0 = 1$ factor is present but trivial). The error: students confuse complex roots $m = \pm 2i$ with the solution form, writing $y = A\mathrm{e}^{2ix} + B\mathrm{e}^{-2ix}$. While mathematically equivalent via Euler's formula, the IB mark scheme requires the real sinusoidal form $y = A\cos 2x + B\sin 2x$.

Question 5 Hard [6 marks]

Find the general solution of the following differential equation.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4\mathrm{e}^{3x}$$

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Complementary function: auxiliary equation $m^2 - 3m + 2 = (m-1)(m-2) = 0$, roots $m = 1, 2$, so $y_c = A\mathrm{e}^x + B\mathrm{e}^{2x}$. Particular integral: try $y_p = k\mathrm{e}^{3x}$. Then $y_p' = 3k\mathrm{e}^{3x}$, $y_p'' = 9k\mathrm{e}^{3x}$. Substitute: $9k\mathrm{e}^{3x} - 9k\mathrm{e}^{3x} + 2k\mathrm{e}^{3x} = 4\mathrm{e}^{3x}$, so $2k = 4$, giving $k = 2$. Particular integral: $y_p = 2\mathrm{e}^{3x}$. General solution: $y = A\mathrm{e}^x + B\mathrm{e}^{2x} + 2\mathrm{e}^{3x}$. The error: students try $y_p = k\mathrm{e}^{3x}$ and correctly find $k = 2$, but then omit y_p from the general solution, writing only y_c. The general solution is always $y = y_c + y_p$. Never omit the particular integral from the final answer.

Question 6 Hard [6 marks]

Find the particular solution of the following differential equation, given that $y = 1$ and $\dfrac{dy}{dx} = 0$ when $x = 0$.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x$$

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Auxiliary equation: $m^2 + 2m + 1 = (m+1)^2 = 0$*. Repeated root* $m = -1$*:* $y_c = (A + Bx)e^{-x}$*. Particular integral: try* $y_p = ax + b$*.* $y_p' = a$*,* $y_p'' = 0$*. Substitute:* $0 + 2a + (ax + b) = x$*, so* $ax + (2a + b) = x$*: compare coefficients:* $a = 1$*,* $2a + b = 0$*, so* $b = -2$*.* $y_p = x - 2$*. General solution:* $y = (A + Bx)e^{-x} + x - 2$*. Apply* $y(0) = 1$*:* $A + 0 - 2 = 1 \Rightarrow A = 3$*. Apply* $y'(0) = 0$*:* $y' = (B - (A + Bx))e^{-x} + 1$*; at* $x = 0$*:* $B - A + 1 = 0$*, so* $B = A - 1 = 2$*. Particular solution:* $y = (3 + 2x)e^{-x} + x - 2$*. The repeated root form* $(A + Bx)e^{-x}$ *is frequently written incorrectly as* $(A + B)e^{-x}$ *or* $Ae^{-x} + Be^{-x}$ *– both wrong. A repeated root* m *always gives* $y_c = (A + Bx)e^{mx}$*.*

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

Identify type	$f(y/x) = v + 1 \Rightarrow$ homogeneous; let $y = vx$	**R1**
Substitute	$v + x\dfrac{dv}{dx} = v + 1 \Rightarrow x\dfrac{dv}{dx} = 1$	**M1**
Separate and integrate	$dv = \dfrac{dx}{x} \Rightarrow v = \ln\lvert x\rvert + C$	**M1**
Back-substitute $v = y/x$	$\dfrac{y}{x} = \ln\lvert x\rvert + C$	**A1**
General solution	$y = x\ln\lvert x\rvert + Cx$	**A1**

Final answer: $y = x\ln|x| + Cx$

Solution – Question 2

Identify type	$\dfrac{dy}{dx} = \dfrac{1+v^2}{2v} \Rightarrow$ homogeneous; let $y = vx$	**R1**
Substitute	$x\dfrac{dv}{dx} = \dfrac{1-v^2}{2v}$	**M1**
Separate	$\dfrac{2v}{1-v^2}\,dv = \dfrac{dx}{x}$	**M1**
Integrate: $\int \frac{2v}{1-v^2}dv = -\ln\lvert 1-v^2\rvert$	$-\ln\lvert 1-v^2\rvert = \ln\lvert x\rvert + C$	**A1**
Rearrange: $1 - v^2 = \frac{B}{x}$	$x^2 - y^2 = Bx$	**A1**

Final answer: $x^2 - y^2 = Bx$

Solution – Question 3

Auxiliary equation	$m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0$	**M1**
Roots	$m = 2,\ m = 3$ (distinct real roots)	**A1**
General solution	$y = Ae^{2x} + Be^{3x}$	**A1**

Final answer: $y = Ae^{2x} + Be^{3x}$

Solution – Question 4

Auxiliary equation	$m^2 + 4 = 0 \;\Rightarrow\; m = \pm 2i$	**M1**
Complex roots $\alpha \pm \beta i$	$\alpha = 0,\ \beta = 2$	**A1**
General solution	$y = A\cos 2x + B\sin 2x$	**A1**

Final answer: $y = A\cos 2x + B\sin 2x$

Solution – Question 5

Auxiliary equation	$m^2 - 3m + 2 = (m-1)(m-2) = 0 \Rightarrow m = 1, 2$	**M1**
Complementary function	$y_c = Ae^x + Be^{2x}$	**A1**
Try $y_p = ke^{3x}$	$9k - 9k + 2k = 4 \Rightarrow k = 2$	**M1**
Particular integral	$y_p = 2e^{3x}$	**A1**
General solution	$y = Ae^x + Be^{2x} + 2e^{3x}$	**A1**

Final answer: $y = Ae^x + Be^{2x} + 2e^{3x}$

Solution – Question 6

Auxiliary equation	$(m+1)^2 = 0 \Rightarrow m = -1$ (repeated)	**M1**
Complementary function	$y_c = (A + Bx)e^{-x}$	**A1**
Try $y_p = ax + b$: substitute	$2a + (ax + b) = x \Rightarrow a = 1, b = -2$	**M1**
Particular integral	$y_p = x - 2$	**A1**
General solution	$y = (A + Bx)e^{-x} + x - 2$	**A1**
Apply $y(0) = 1$	$A - 2 = 1 \Rightarrow A = 3$	**A1**
Apply $y'(0) = 0$: $y' = (B - (A + Bx))e^{-x} + 1$	$B - A + 1 = 0 \Rightarrow B = 2$	**A1**
Particular solution	$y = (3 + 2x)e^{-x} + x - 2$	**A1**

Final answer: $y = (3 + 2x)e^{-x} + x - 2$
