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Loci & Argand Diagrams

Recognition Training – Set III

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 1 – Number & Algebra
Level	Medium $\to$ Hard
Questions	6
Total marks	32
Instructions	Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark. Sketch loci clearly, labelling centres, radii, and key points.

BEFORE YOU BEGIN

A **locus** in the Argand diagram is the set of points z satisfying a given condition.

$|z - a| = r$: circle with centre a and radius r.

$|z - a| = |z - b|$: perpendicular bisector of the segment joining a and b.

$\arg(z - a) = \theta$: half-line from a (not including a) at angle θ to the positive real axis.

$|z - a| \leq r$: closed disc (interior and boundary of the circle).

When two loci intersect, solve the equations simultaneously by substituting $z = x + iy$.

Question 1 — Medium — [4 marks]

Describe and sketch the locus of z satisfying $|z - 3 + 2i| = 4$.

MISTAKE ANALYSIS

Write $|z - (3 - 2i)| = 4$: circle with centre $3 - 2i$ (i.e. point $(3, -2)$ in the Argand diagram) and radius 4. Students who write the centre as $(3, 2)$ have misread $-3 + 2i$ as the centre without correctly identifying the complex number subtracted from z. The condition $|z - w| = r$ has centre w, not $-w$. Here $w = 3 - 2i$ (note: $z - 3 + 2i = z - (3 - 2i)$). The centre is the complex number $3 - 2i$, plotted at $(3, -2)$.

Question 2 Medium [5 marks]

Describe the locus of z satisfying $|z-2| = |z+4i|$, and find the Cartesian equation of this locus.

MISTAKE ANALYSIS

This is the perpendicular bisector of the segment from 2 *(i.e.* $(2,0)$*) to* $-4i$ *(i.e.* $(0,-4)$*). Let* $z = x+iy$*.* $|z-2|^2 = (x-2)^2+y^2$ *and* $|z+4i|^2 = x^2+(y+4)^2$*. Setting equal:* $x^2-4x+4+y^2 = x^2+y^2+8y+16$*.* $-4x+4 = 8y+16$*, so* $-4x-8y = 12$*, giving* $x+2y+3 = 0$*. Students who interpret* $|z-2| = |z+4i|$ *as a circle rather than a line are confusing this with* $|z-a| = r$*. Whenever the condition equates two moduli* $|z-a| = |z-b|$*, the locus is a perpendicular bisector – a straight line, not a circle.*

Question 3 Medium [5 marks]

Sketch the locus of z satisfying $\arg(z-1-i) = \dfrac{\pi}{4}$, and find the Cartesian equation of the locus, stating clearly which part of the line is included.

MISTAKE ANALYSIS

This is a half-line starting at $1+i$ *(not including* $1+i$*) at angle* $\pi/4$ *to the positive real direction. Let* $z = x+iy$*. Then* $z-1-i = (x-1)+i(y-1)$*.* $\arg((x-1)+i(y-1)) = \pi/4$ *means* $\dfrac{y-1}{x-1} = \tan(\pi/4) = 1$*, so* $y-1 = x-1$*, i.e.* $y = x$*. Restriction:* $x-1 > 0$ *(the argument* $\pi/4$ *is in the first quadrant, so* $\mathrm{Re}(z-1-i) > 0$*), meaning* $x > 1$*. Locus: the ray* $y = x$ *for* $x > 1$*, not including the point* $(1,1)$*. Students who draw the full line* $y = x$ *rather than the half-line* $x > 1$ *lose the mark for the restriction – the argument condition specifies a direction, and the half-line starts at but excludes the fixed point.*

Question 4 Medium–Hard [6 marks]

Find the Cartesian equation of the locus of z satisfying $\left|\dfrac{z-1}{z+1}\right| = 2$.

MISTAKE ANALYSIS

Let $z = x+iy$*.* $|z-1|^2 = (x-1)^2+y^2$*,* $|z+1|^2 = (x+1)^2+y^2$*.* $\left|\dfrac{z-1}{z+1}\right| = 2 \Rightarrow |z-1|^2 = 4|z+1|^2$*.* $(x-1)^2+y^2 = 4((x+1)^2+y^2)$*.* $x^2-2x+1+y^2 = 4x^2+8x+4+4y^2$*.* $3x^2+3y^2+10x+3 = 0$*.*

$x^2+y^2+\dfrac{10}{3}x+1=0$. Complete the square: $(x+\frac{5}{3})^2+y^2=\dfrac{25}{9}-1=\dfrac{16}{9}$. Circle: centre $\left(-\dfrac{5}{3},0\right)$, radius $\dfrac{4}{3}$. Students who square only one side – writing $|z-1|=2|z+1|$ and then squaring only the left – miss the factor of 4 on the right side. The condition $|z-1|=2|z+1|$ must be squared completely on both sides.

Question 5 Hard [6 marks]

The locus C_1 is the circle $|z-2|=2$ and the locus C_2 is the line $\mathrm{Re}(z)=3$.

(a) Sketch both loci on the same Argand diagram.

(b) Find the points of intersection of C_1 and C_2.

(c) Shade the region satisfying $|z-2|\leq 2$ and $\mathrm{Re}(z)\geq 3$.

MISTAKE ANALYSIS

(a) C_1: circle centre $(2,0)$, radius 2. C_2: vertical line $x=3$. (b) On C_2: $x=3$, so $|z-2|=|1+iy|=\sqrt{1+y^2}=2$, giving $y^2=3$, $y=\pm\sqrt{3}$. Intersection points: $3+i\sqrt{3}$ and $3-i\sqrt{3}$. (c) $|z-2|\leq 2$: interior and boundary of C_1. $\mathrm{Re}(z)\geq 3$: right-hand side of C_2. The shaded region is the part of the closed disc C_1 to the right of the line $x=3$: a circular segment (the minor arc between $3\pm i\sqrt{3}$ and the line $x=3$). The error: students who shade the region to the left of $x=3$ have reversed the inequality direction. $\mathrm{Re}(z)\geq 3$ means $x\geq 3$, which is to the right of the line.

Question 6 Hard [6 marks]

Find the maximum value of $|z|$ for complex numbers z satisfying $|z-3-4i|\leq 2$.

MISTAKE ANALYSIS

$|z-3-4i|\leq 2$ is the closed disc with centre $w=3+4i$ and radius 2. $|w|=\sqrt{9+16}=5$. By the triangle inequality: $|z|\leq|z-w|+|w|\leq 2+5=7$. Equality holds when z, w, and 0 are collinear with z on the far side of w from 0, i.e. $z=w+2\cdot\dfrac{w}{|w|}=3+4i+\dfrac{2(3+4i)}{5}=3+4i+\frac{6}{5}+\frac{8}{5}i=\dfrac{21}{5}+\dfrac{28}{5}i$. Maximum $|z|=7$. Students who compute $|w|+2=5+2=7$ without invoking the triangle inequality get the right answer but earn no method mark. The triangle inequality $|z|\leq|z-w|+|w|$ must be stated explicitly, with the equality condition identified.

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

Rewrite	$\lvert z-(3-2i)\rvert = 4$	**M1**
Locus	Circle, centre $3-2i=(3,-2)$, radius 4	**A1**

Final answer: Circle with centre $(3,-2)$ and radius 4

Solution – Question 2

Set $z = x + iy$; square both sides	$(x-2)^2+y^2=x^2+(y+4)^2$	**M1**
Expand and simplify	$-4x+4=8y+16 \;\Rightarrow\; x+2y+3=0$	**A1**
Locus	Perpendicular bisector of segment from $(2,0)$ to $(0,-4)$	**R1**

Final answer: $x+2y+3=0$

Solution – Question 3

$\arg((x-1)+i(y-1))=\pi/4$	$\dfrac{y-1}{x-1}=\tan\dfrac{\pi}{4}=1$	**M1**
Equation	$y=x$	**A1**
Restriction: $\mathrm{Re}(z-1-i)>0$	$x>1$ (half-line only, excluding $(1,1)$)	**R1**

Final answer: $y=x, \quad x>1$ (point $(1,1)$ excluded)

Solution – Question 4

$\lvert z-1\rvert^2 = 4\lvert z+1\rvert^2$	$(x-1)^2+y^2 = 4((x+1)^2+y^2)$	**M1**
Expand	$3x^2+3y^2+10x+3=0$	**M1**
Complete square	$\left(x+\frac{5}{3}\right)^2+y^2=\frac{16}{9}$	**A1**
Locus	Circle, centre $\left(-\frac{5}{3},0\right)$, radius $\frac{4}{3}$	**A1**

Final answer: $\left(x+\frac{5}{3}\right)^2+y^2=\frac{16}{9}$

Solution – Question 5

(a) C_1: circle centre $(2,0)$, radius 2. C_2: vertical line $x=3$.

(b)

On C_2: $x=3$; substitute into $\lvert z-2\rvert=2$	$\sqrt{1+y^2}=2 \Rightarrow y=\pm\sqrt{3}$	**M1**
Intersection points	$3+i\sqrt{3}$ and $3-i\sqrt{3}$	**A1**

(c) Shade the closed region inside C_1 with $x \geq 3$: the minor circular segment.

Final answer: (b) $3 \pm i\sqrt{3}$

Solution – Question 6

Centre $w=3+4i$, $\lvert w\rvert=5$, radius 2		**R1**
Triangle inequality	$\lvert z\rvert \leq \lvert z-w\rvert+\lvert w\rvert \leq 2+5=7$	**M1**
Equality when z, w, 0 collinear	$z=w+\frac{2w}{\lvert w\rvert}=\frac{21}{5}+\frac{28}{5}i$	**A1**

Final answer: $\lvert z\rvert_{\max}=7$

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