De Moivre's Theorem & Roots of Unity

Recognition Training – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 1 – Number & Algebra
Level	Medium → Hard
Questions	6
Total marks	32
Instructions	Show all working. **M1** = method mark. **A1** = accuracy mark. **R1** = reasoning mark. Do not use a calculator unless stated.

BEFORE YOU BEGIN

De Moivre's Theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all $n \in \mathbb{Z}$.

Powers: If $z = r(\cos\theta + i\sin\theta)$ then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Roots: The nth roots of unity are $e^{2\pi ik/n} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$ for $k = 0, 1, \ldots, n-1$. The nth roots of a complex number $z = re^{i\theta}$ are $r^{1/n}e^{i(\theta+2\pi k)/n}$ for $k = 0, 1, \ldots, n-1$.

The key application of De Moivre: expand $(\cos\theta + i\sin\theta)^n$ using the binomial theorem, equate real and imaginary parts to express $\cos n\theta$ and $\sin n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

Question 1 — Medium — [4 marks]

Use De Moivre's theorem to find $(1+i)^8$.

MISTAKE ANALYSIS

Write $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. $(1+i)^8 = (\sqrt{2})^8\left(\cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4}\right) = 16(\cos 2\pi + i\sin 2\pi) = 16(1+0) = 16$. *Students who expand* $(1+i)^8$ *by repeated squaring are correct but take longer.* $(1+i)^2 = 2i$; $(2i)^2 = -4$; $(-4)^2 = 16$ *– gives the same answer but involves tracking signs through three squarings, each with risk of sign error. De Moivre is faster and less error-prone for high powers.*

Question 2 Medium [5 marks]

Use De Moivre's theorem and the binomial expansion of $(\cos\theta + i\sin\theta)^3$ to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

MISTAKE ANALYSIS

$(\cos\theta+i\sin\theta)^3 = \cos^3\theta+3\cos^2\theta(i\sin\theta)+3\cos\theta(i\sin\theta)^2+(i\sin\theta)^3 = \cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$. *Real part:* $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$. *Imaginary part:* $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$. *Students who write* $(i\sin\theta)^2 = i^2\sin^2\theta = \sin^2\theta$ *(forgetting* $i^2 = -1$*) get* $+3\cos\theta\sin^2\theta$ *in the real part – the wrong sign. Track* $i^2 = -1$, $i^3 = -i$ *explicitly at every step.*

Question 3 Medium [5 marks]

Find all cube roots of $8i$, giving your answers in the form $a + bi$.

MISTAKE ANALYSIS

Write $8i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. *Cube roots:* $8^{1/3} = 2$, *angles* $\frac{\pi/2 + 2\pi k}{3}$ *for* $k = 0, 1, 2$. $k = 0$: $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{3} + i$. $k = 1$: $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\sqrt{3} + i$. $k = 2$: $2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2i$. *Students who find only one root* ($\sqrt[3]{8i} = \sqrt[3]{8}\cdot\sqrt[3]{i}$ *and estimate* $\sqrt[3]{i}$) *miss the other two. Every complex number has exactly* n *distinct* n*th roots. The three roots are equally spaced at angles of* $2\pi/3$ *apart.*

Question 4 Medium–Hard [6 marks]

(a) Write down the three cube roots of unity in the form $\cos\theta + i\sin\theta$.

(b) Show that if ω is a primitive cube root of unity then $1 + \omega + \omega^2 = 0$.

(c) Hence find the value of $(1 + \omega^2)(1 + \omega^4)$.

MISTAKE ANALYSIS

(a) 1, $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$. *(b) The cube roots are the roots of* $z^3 - 1 = 0$, *factored as* $(z-1)(z^2+z+1) = 0$. *For* $z \neq 1$: $\omega^2 + \omega + 1 = 0$, *so* $1 + \omega + \omega^2 = 0$. ✓ *(c)* $\omega^4 = \omega^3\cdot\omega = 1\cdot\omega = \omega$. $(1+\omega^2)(1+\omega) = 1 + \omega + \omega^2 + \omega^3 = 0 + 1 = 1$. *Students who compute* $(1+\omega^2)(1+\omega^4)$ *without reducing* $\omega^4 = \omega$ *first write* $(1+\omega^2)(1+\omega^4)$ *and expand without simplifying – the result is correct but much harder. Always reduce high powers of* ω *using* $\omega^3 = 1$.

Question 5 Hard [6 marks]

(a) Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(b) Hence solve $16x^5 - 20x^3 + 5x = 1$ for $x \in [-1, 1]$.

MISTAKE ANALYSIS

(a) Expand $(\cos\theta + i\sin\theta)^5$ using the binomial theorem and equate real parts. The real part gives $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$. Use $\sin^2\theta = 1 - \cos^2\theta$: $= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. ✓ (b) The equation is $\cos 5\theta = 1$ with $x = \cos\theta$. $5\theta = 2\pi k$, so $\theta = \dfrac{2\pi k}{5}$ for $k = 0, 1, 2, 3, 4$. $x = \cos\dfrac{2\pi k}{5}$: $x = 1$, $\cos\dfrac{2\pi}{5}$, $\cos\dfrac{4\pi}{5}$, $\cos\dfrac{6\pi}{5} = \cos\dfrac{4\pi}{5}$, $\cos\dfrac{8\pi}{5} = \cos\dfrac{2\pi}{5}$. Distinct values: $x = 1$, $\cos\dfrac{2\pi}{5}$, $\cos\dfrac{4\pi}{5}$. But the polynomial has degree 5 – students must account for all 5 roots. Since $\cos(2\pi - \theta) = \cos\theta$: three distinct values, each giving one or two θ values. The 5 roots in $[-1, 1]$ are: 1, $\cos\frac{2\pi}{5}$, $\cos\frac{4\pi}{5}$, $\cos\frac{6\pi}{5}(=\cos\frac{4\pi}{5})$, $\cos\frac{8\pi}{5}(=\cos\frac{2\pi}{5})$. So as polynomial roots: $x = 1$ (once), $x = \cos\frac{2\pi}{5}$ (twice), $x = \cos\frac{4\pi}{5}$ (twice).

Question 6 Hard [6 marks]

A polynomial $p(z) = z^4 + az^3 + bz^2 + cz + d$ has real coefficients. Given that $2 + 3i$ is a root of $p(z)$, find all roots of $p(z)$ and determine the values of a, b, c, d if the product of all roots equals 13.

MISTAKE ANALYSIS

Since $p(z)$ has real coefficients, $\overline{2+3i} = 2 - 3i$ is also a root. The product $(z-(2+3i))(z-(2-3i)) = z^2 - 4z + 13$. Since $p(z)$ has degree 4, the remaining factor is $(z-r)(z-s)$ for real roots r, s. Product of all roots $= r \cdot s \cdot (2+3i)(2-3i) = rs \cdot 13 = 13$, so $rs = 1$. Without additional information, many such r, s are possible (e.g. $r = s = 1$ or $r = 2, s = 1/2$). With $r = s = 1$: $p(z) = (z^2-4z+13)(z-1)^2 = (z^2-4z+13)(z^2-2z+1)$. Expand: $a = -6$, $b = 20$, $c = -30$, $d = 13$. The critical point: a polynomial with real coefficients always has complex roots appearing in conjugate pairs. Students who take $2+3i$ as the only complex root and look for three real roots misunderstand this fundamental property.

M1 = method mark. **A1** = accuracy mark. **R1** = reasoning mark.

Solution – Question 1

Modulus-arg form	$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$	**M1**
De Moivre	$(1+i)^8=(\sqrt{2})^8(\cos 2\pi+i\sin 2\pi)=16(1)$	**M1**

Final answer: 16

Solution – Question 2

Expand $(\cos\theta+i\sin\theta)^3$	$\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$	**M1**
Real part	$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta=4\cos^3\theta-3\cos\theta$	**A1**
Imaginary part	$\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta=3\sin\theta-4\sin^3\theta$	**A1**

Final answer: $\cos 3\theta=4\cos^3\theta-3\cos\theta;\quad \sin 3\theta=3\sin\theta-4\sin^3\theta$

Solution – Question 3

Write $8i=8(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2})$	$r^{1/3}=2;\quad$ angles $\dfrac{\pi/2+2\pi k}{3}$	**M1**
$k=0$: $\theta=\pi/6$	$2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})=\sqrt{3}+i$	**A1**
$k=1$: $\theta=5\pi/6$	$2(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})=-\sqrt{3}+i$	**A1**
$k=2$: $\theta=3\pi/2$	$2(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2})=-2i$	**A1**

Final answer: $\sqrt{3}+i,\quad -\sqrt{3}+i,\quad -2i$

Solution – Question 4

(a) Three cube roots of unity	$1, \quad \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \quad \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$	**A1**
(b) $z^3 - 1 = (z-1)(z^2+z+1) = 0$	$z^2 + z + 1 = 0 \Rightarrow \omega^2 + \omega + 1 = 0 \Rightarrow 1 + \omega + \omega^2 = 0$ ✓	**R1**
(c) Reduce: $\omega^4 = \omega^3 \cdot \omega = \omega$	$(1+\omega^2)(1+\omega) = 1 + \omega + \omega^2 + \omega^3 = 0 + 1$	**M1**

Final answer: $(1+\omega^2)(1+\omega^4) = 1$

Solution – Question 5

Part (a):

Real part of $(\cos\theta + i\sin\theta)^5$	$\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$	**M1**
Use $\sin^2\theta = 1 - \cos^2\theta$	$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ ✓	**A1**

Part (b):

Equation is $\cos 5\theta = 1$; set $x = \cos\theta$	$5\theta = 2\pi k, \quad k = 0, 1, 2, 3, 4$	**M1**
Distinct x values	$x = 1, \quad \cos\frac{2\pi}{5}, \quad \cos\frac{4\pi}{5}$	**A1**

Final answer: $x = 1, \quad \cos\frac{2\pi}{5}, \quad \cos\frac{4\pi}{5}$ (each with multiplicity from $5\theta = 2\pi k$)

Solution – Question 6

Conjugate root	$2 - 3i$ is also a root (real coefficients)	**R1**
Quadratic factor	$(z - (2+3i))(z - (2-3i)) = z^2 - 4z + 13$	**M1**
Product of all roots $= 13$	$(2+3i)(2-3i) \cdot rs = 13 \cdot rs = 13 \Rightarrow rs = 1$	**M1**
Take $r = s = 1$: remaining factor	$(z-1)^2$	**A1**
Expand $p(z)$	$(z^2 - 4z + 13)(z^2 - 2z + 1) \Rightarrow a = -6, b = 20, c = -30, d = 13$	**A1**

Final answer: Roots: $2+3i,\ 2-3i,\ 1,\ 1; \quad a = -6,\ b = 20,\ c = -30,\ d = 13$
