

Cartesian Form & Complex Operations

Recognition Training – Set I

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 1 – Number & Algebra
Level	Easy → Medium
Questions	6
Total marks	30
Instructions	Show all working. M1 = method mark. A1 = accuracy mark. R1 = reasoning mark. Do not use a calculator unless stated. Write answers in the form $a + bi$ unless otherwise specified.

BEFORE YOU BEGIN

A complex number in Cartesian form is $z = x + iy$ where $x = \operatorname{Re}(z)$ and $y = \operatorname{Im}(z)$. The **modulus** is $|z| = \sqrt{x^2 + y^2}$ and the **argument** is $\arg(z) = \theta$ where $\cos \theta = \frac{x}{|z|}$ and $\sin \theta = \frac{y}{|z|}$, with $\theta \in (-\pi, \pi]$.

The **conjugate** is $\bar{z} = x - iy$. Key properties: $z\bar{z} = |z|^2$ and $\frac{z}{z_1 z_2} = \frac{\bar{z} \bar{z}_1 \bar{z}_2}{\bar{z}_1 \bar{z}_2}$. To divide complex numbers, multiply numerator and denominator by the conjugate of the denominator.

Question 1

Easy

[4 marks]

Let $z_1 = 3 + 4i$ and $z_2 = 1 - 2i$.

- (a) Find $z_1 + z_2$ and $z_1 - z_2$.
- (b) Find $|z_1|$ and $|z_2|$.

MISTAKE ANALYSIS

$z_1 + z_2 = (3+1) + (4-2)i = 4 + 2i$. $z_1 - z_2 = (3-1) + (4+2)i = 2 + 6i$. $|z_1| = \sqrt{9+16} = 5$. $|z_2| = \sqrt{1+4} = \sqrt{5}$. *Students who compute $|z_1 + z_2| = |4 + 2i| = \sqrt{20} = 2\sqrt{5}$ and then claim $|z_1 + z_2| = |z_1| + |z_2| = 5 + \sqrt{5}$ are confusing the modulus of a sum with the sum of moduli. These are not equal in general. The triangle inequality states $|z_1 + z_2| \leq |z_1| + |z_2|$, with equality only when z_1 and z_2 have the same argument.*

Question 2

Easy

[4 marks]

Let $z = 2 + 3i$.

- (a) Find z^2 .
- (b) Find $z\bar{z}$ and verify it equals $|z|^2$.

MISTAKE ANALYSIS

$z^2 = (2 + 3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i$. $z\bar{z} = (2 + 3i)(2 - 3i) = 4 - 9i^2 = 4 + 9 = 13$. $|z|^2 = 4 + 9 = 13$. ✓ *Students who compute $z^2 = (2 + 3i)^2 = 4 + 9i^2 = -5$ (forgetting the cross term $2 \cdot 2 \cdot 3i = 12i$) are applying the wrong squaring formula. $(a + b)^2 = a^2 + 2ab + b^2$ - the middle term $2ab$ is essential. Also: $i^2 = -1$, not $+1$. Check every appearance of i^2 .*

Question 3

Easy-Medium

[5 marks]

Find $\frac{3 + 4i}{1 + 2i}$, giving your answer in the form $a + bi$.**MISTAKE ANALYSIS**

Multiply by $\frac{1 - 2i}{1 - 2i}$: $\frac{(3 + 4i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{3 - 6i + 4i - 8i^2}{1 + 4} = \frac{3 - 2i + 8}{5} = \frac{11 - 2i}{5} = \frac{11}{5} - \frac{2}{5}i$. *Students who multiply numerator and denominator by $(1 + 2i)$ instead of $(1 - 2i)$ make the denominator $(1 + 2i)^2 = 1 + 4i - 4 = -3 + 4i$ - complex, not real. The conjugate of $1 + 2i$ is $1 - 2i$. Multiplying by the conjugate is what clears the denominator.*

Question 4

Medium

[5 marks]

Write $z = -\sqrt{3} + i$ in modulus-argument form $r(\cos \theta + i \sin \theta)$, where $\theta \in (-\pi, \pi]$.**MISTAKE ANALYSIS**

$r = |z| = \sqrt{3 + 1} = 2$. $\cos \theta = -\sqrt{3}/2$, $\sin \theta = 1/2$. *Since $\text{Re}(z) < 0$ and $\text{Im}(z) > 0$: second quadrant.*
 $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$. $z = 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$. *Students who compute $\theta = \arctan\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}$ without adjusting for the quadrant are in the fourth quadrant, not the second. \arctan returns values in $(-\pi/2, \pi/2)$; when $\text{Re}(z) < 0$, add or subtract π to place θ in the correct quadrant.*

Question 5

Medium

[6 marks]

Solve the equation $z^2 + 4z + 13 = 0$, giving your answers in the form $a + bi$.

MISTAKE ANALYSIS

Discriminant: $\Delta = 16 - 52 = -36$. $z = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i$. Students who write $\sqrt{-36} = \pm 6$ (dropping the i) produce real roots $z = -2 \pm 3$ giving $z = 1$ or $z = -5$ – which do not satisfy the equation. $\sqrt{-36} = \sqrt{36} \cdot \sqrt{-1} = 6i$. Always write $\sqrt{-k} = i\sqrt{k}$ for $k > 0$ before substituting. Verify: $(-2 + 3i)^2 + 4(-2 + 3i) + 13 = 4 - 12i - 9 - 8 + 12i + 13 = 0$. ✓

Question 6

Medium

[6 marks]

The complex number z satisfies $z + \frac{2}{\bar{z}} = 3$. Find the possible values of z .

MISTAKE ANALYSIS

Let $z = x + iy$, so $\bar{z} = x - iy$ and $\frac{2}{\bar{z}} = \frac{2(x + iy)}{x^2 + y^2}$. The equation becomes: $x + iy + \frac{2x}{x^2 + y^2} + \frac{2iy}{x^2 + y^2} = 3$. Equate real and imaginary parts: Real: $x + \frac{2x}{x^2 + y^2} = 3$. Imaginary: $y + \frac{2y}{x^2 + y^2} = 0$, so $y\left(1 + \frac{2}{x^2 + y^2}\right) = 0$. Since $1 + \frac{2}{x^2 + y^2} > 0$ always: $y = 0$. With $y = 0$: $x + \frac{2}{x} = 3$, so $x^2 - 3x + 2 = 0$, giving $x = 1$ or $x = 2$. Solutions: $z = 1$ or $z = 2$ (both real). The error: students who multiply both sides by \bar{z} write $z\bar{z} + 2 = 3\bar{z}$, i.e. $|z|^2 + 2 = 3\bar{z}$. Since $|z|^2$ is real, taking the imaginary part of the right side: $\text{Im}(3\bar{z}) = -3y = 0$, giving $y = 0$ directly. Both routes work.

WORKED SOLUTIONS – SET I – CARTESIAN FORM & COMPLEX OPERATIONS

M1 = method mark. A1 = accuracy mark. R1 = reasoning mark.

Solution – Question 1

$$(a) z_1 + z_2 \quad (3 + 1) + (4 - 2)i = 4 + 2i \quad \mathbf{A1}$$

$$z_1 - z_2 \quad (3 - 1) + (4 + 2)i = 2 + 6i \quad \mathbf{A1}$$

$$(b) |z_1| \quad \sqrt{3^2 + 4^2} = \sqrt{25} = 5 \quad \mathbf{A1}$$

$$|z_2| \quad \sqrt{1^2 + 2^2} = \sqrt{5} \quad \mathbf{A1}$$

Final answer: $z_1 + z_2 = 4 + 2i$; $z_1 - z_2 = 2 + 6i$; $|z_1| = 5$; $|z_2| = \sqrt{5}$

Solution – Question 2

$$(a) z^2 = (2 + 3i)^2 \quad 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i \quad \mathbf{M1}$$

$$(b) z\bar{z} = (2 + 3i)(2 - 3i) \quad 4 - 9i^2 = 4 + 9 = 13 \quad \mathbf{M1}$$

$$\text{Verify} \quad |z|^2 = 2^2 + 3^2 = 13 \checkmark \quad \mathbf{R1}$$

Final answer: $z^2 = -5 + 12i$; $z\bar{z} = 13 = |z|^2$

Solution – Question 3

$$\text{Multiply by con-} \quad \frac{(3 + 4i)(1 - 2i)}{(1 + 2i)(1 - 2i)} \quad \mathbf{M1}$$

$$\text{jugate} \quad \text{Denominator} \quad 1^2 + 2^2 = 5 \quad \mathbf{A1}$$

$$\text{Numerator} \quad 3 - 6i + 4i - 8i^2 = 3 - 2i + 8 = 11 - 2i \quad \mathbf{A1}$$

$$\text{Divide} \quad \frac{11 - 2i}{5} = \frac{11}{5} - \frac{2}{5}i \quad \mathbf{A1}$$

Final answer: $\frac{11}{5} - \frac{2}{5}i$

Solution – Question 4

Modulus	$r = \sqrt{3+1} = 2$	A1
Identify quadrant	$\text{Re}(z) < 0, \text{Im}(z) > 0 \Rightarrow$ second quadrant	R1
$\cos \theta = -\frac{\sqrt{3}}{2},$ $\sin \theta = \frac{1}{2}$	$\theta = \frac{5\pi}{6}$	A1
Modulus-argument form	$2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$	A1
Final answer:	$2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$	

Solution – Question 5

Quadratic formula	for- $\Delta = 16 - 52 = -36$	M1
$\sqrt{-36} = 6i$	$z = \frac{-4 \pm 6i}{2}$	A1
Solutions	$z = -2 + 3i$ or $z = -2 - 3i$	A1

Final answer: $z = -2 \pm 3i$

Solution – Question 6

Imaginary part of equation	$y\left(1 + \frac{2}{x^2 + y^2}\right) = 0 \Rightarrow y = 0$	M1
Real part with $y = 0$	$x + \frac{2}{x} = 3 \Rightarrow x^2 - 3x + 2 = 0$	M1
Factorise	$(x - 1)(x - 2) = 0 \Rightarrow x = 1$ or $x = 2$	A1

Final answer: $z = 1$ or $z = 2$
