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Applications of Integration

Mistake Analysis – Set II

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 5 – Calculus: Applications of Integration
Level	Medium → Hard
Questions	6
Marks	36 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Washer method: $V = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2\right) dx$ (outer radius squared minus inner radius squared).

Rotation about y-axis (disk method): express x as a function of y and integrate $V = \pi \int_c^d x^2\, dy$.

Shell method: $V = 2\pi \int_a^b x\, f(x)\, dx$ (rotation about y-axis).

Average value: $\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$.

Integrating with respect to y: when a curve is more naturally expressed as $x = g(y)$, integrate horizontally.

Question 1

Medium [5 marks]

The region enclosed by $y = x^2$ and $y = x$ is rotated 360 about the x-axis. Find the volume using the washer method.

MISTAKE ANALYSIS

The outer radius is $y = x$ and inner radius is $y = x^2$ (since $x \geq x^2$ on $[0,1]$). $V = \pi \int_0^1 \left[(x)^2 - (x^2)^2\right] dx = \pi \int_0^1 (x^2 - x^4)\, dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$. Students who use the disk formula with just one curve get the volume of the solid formed by rotating one boundary, not the region between them. The washer formula requires both radii: $\pi \int (R^2 - r^2)\, dx$ where R is the outer and r the inner radius.

Question 2 Medium [5 marks]

The region bounded by $y = \sqrt{x}$, the y-axis, and $y = 2$ is rotated 360 about the y-axis. Find the volume.

MISTAKE ANALYSIS

Express in terms of y: $y = \sqrt{x} \Rightarrow x = y^2$. *Disk method about y-axis:* $V = \pi \int_0^2 (y^2)^2\,dy = \pi \int_0^2 y^4\,dy = \pi \left[\frac{y^5}{5}\right]_0^2 = \frac{32\pi}{5}$. *The limits are y-values* ($y = 0$ *to* $y = 2$)*, not x-values. Students who integrate with respect to* x *(using limits* $x = 0$ *to* $x = 4$*) are rotating the region the wrong way. When rotating about the y-axis, the natural variable is* y*: express* x *in terms of* y*.*

Question 3 Medium [5 marks]

Find the area of the region bounded by $y = e^x$, $y = 1$, and $x = 2$.

MISTAKE ANALYSIS

$y = e^x$ *and* $y = 1$ *intersect where* $e^x = 1 \Rightarrow x = 0$. *The region is bounded by* $x = 0$ *(left),* $x = 2$ *(right),* $y = 1$ *(bottom),* $y = e^x$ *(top).* $A = \int_0^2 (e^x - 1)\,dx = \left[e^x - x\right]_0^2 = (e^2 - 2) - (1 - 0) = e^2 - 3$. *Students who integrate* e^x *alone (forgetting to subtract the base* $y = 1$*) find the area under* $y = e^x$ *from 0 to 2, which includes the rectangle of area 2 below* $y = 1$*. The area between the curves requires subtracting the lower boundary.*

Question 4 Medium [6 marks]

Find the area enclosed between $y = x^2 - 4$ and $y = 4 - x^2$.

MISTAKE ANALYSIS

Intersections: $x^2 - 4 = 4 - x^2 \Rightarrow 2x^2 = 8 \Rightarrow x = \pm 2$. *On* $[-2, 2]$*:* $4 - x^2 \geq x^2 - 4$*, so upper curve is*

$y = 4 - x^2$. $A = \displaystyle\int_{-2}^{2} [(4-x^2)-(x^2-4)]\,dx = \int_{-2}^{2} (8-2x^2)\,dx = \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$. $= (16 - \frac{16}{3}) - (-16 + \frac{16}{3}) = 32 - \frac{32}{3} = \frac{64}{3}$. *By symmetry:* $2\int_0^2 (8-2x^2)\,dx = 2\left[8x - \frac{2x^3}{3}\right]_0^2 = 2(16 - \frac{16}{3}) = \frac{64}{3}$. *Students who compute the area as* $\int_{-2}^{2} |x^2-4|\,dx$ *are finding a different region. The two parabolas enclose the lens-shaped region between them; integrate top minus bottom.*

Question 5

Hard

[5 marks]

Find the average value of $f(x) = x^2 + 1$ on $[0, 3]$.

MISTAKE ANALYSIS

Average value $= \dfrac{1}{3-0}\displaystyle\int_0^3 (x^2+1)\,dx = \frac{1}{3}\left[\frac{x^3}{3} + x\right]_0^3 = \frac{1}{3}(9+3) = \frac{12}{3} = 4$. *The formula is* $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$. *Students who evaluate* $\frac{f(a)+f(b)}{2} = \frac{1+10}{2} = 5.5$ *(the trapezoidal approximation) are not computing the average value of a function – they are approximating it for a linear function. The average value requires integration.*

Question 6

Hard

[10 marks]

Find the area enclosed between the parabola $y^2 = x$ and the line $y = x - 2$.

MISTAKE ANALYSIS

Express everything in terms of y. *From* $y = x - 2$: $x = y + 2$. *From* $y^2 = x$: $x = y^2$. *Intersections:* $y^2 = y + 2 \Rightarrow y^2 - y - 2 = (y-2)(y+1) = 0 \Rightarrow y = -1$ *or* $y = 2$. *Corresponding* x: $y = -1 \Rightarrow x = 1$; $y = 2 \Rightarrow x = 4$. *On* $[-1, 2]$ *(in* y*): right boundary is* $x = y + 2$, *left is* $x = y^2$. $A = \displaystyle\int_{-1}^{2} [(y+2) - y^2]\,dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3}\right]_{-1}^{2}$. *At* $y = 2$: $2 + 4 - 8/3 = 6 - 8/3 = 10/3$. *At* $y = -1$: $1/2 - 2 + 1/3 = -7/6$. $A = \frac{10}{3} + \frac{7}{6} = \frac{20}{6} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2}$. *Students who try to integrate with respect to* x *must split the parabola into* $y = \sqrt{x}$ *and* $y = -\sqrt{x}$, *giving two integrals. Integrating with respect to* y *treats it as a single integral.*

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

Washer: $\pi \int_0^1 (x^2 - x^4)\,dx$ $[\frac{x^3}{3} - \frac{x^5}{5}]_0^1 = \frac{2\pi}{15}$ **M1**

Solution – Question 2

$x = y^2;\ V = \pi \int_0^2 y^4\,dy$ $[\frac{y^5}{5}]_0^2 = \frac{32\pi}{5}$ **M1**

Solution – Question 3

Intersect at $x = 0;\ A = \int_0^2 (e^x - 1)\,dx = [e^x - x]_0^2$ $e^2 - 3$ **M1**

Solution – Question 4

Intersect at $x = \pm 2;\ A = \int_{-2}^2 (8 - 2x^2)\,dx = [8x - \frac{2x^3}{3}]_{-2}^2$ $\frac{64}{3}$ **M1**

Solution – Question 5

$\bar{f} = \frac{1}{3}\int_0^3 (x^2 + 1)\,dx = \frac{1}{3}[12]$ 4 **M1**

Solution – Question 6

$x = y^2$, $x = y+2$; intersect $y = -1, 2$; $A = \int_{-1}^{2} [(y+2) - y^2]\, dy$ **M1**

$= [\frac{y^2}{2} + 2y - \frac{y^3}{3}]_{-1}^{2} = \frac{10}{3} + \frac{7}{6}$ $\frac{9}{2}$ **A1**