# Applications of Integration

*Mistake Analysis – Set III*

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| **Course** | IB Mathematics: Analysis & Approaches HL |
| **Topic** | Topic 5 – Calculus: Applications of Integration |
| **Level** | Medium → Hard |
| **Questions** | 6 |
| **Marks** | 36 total. **M1** method · **A1** accuracy · **R1** reasoning. |

## BEFORE YOU BEGIN

**From acceleration to velocity:** $v(t) = \int a(t)\,dt$; use the initial condition $v(0)$ to find $C$.

**From velocity to position:** $x(t) = \int v(t)\,dt$; use $x(0)$ to find $C$.

**Average value:** $\bar{f} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$.

**Shell method:** $V = 2\pi \displaystyle\int_a^b x\,f(x)\,dx$ – the circumference-times-height formula for thin cylindrical shells.

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**Question 1** Medium [6 marks]

A particle has acceleration $a(t) = 6t - 2\,\mathrm{m\,s^{-2}}$ and initial velocity $v(0) = 3\,\mathrm{m\,s^{-1}}$. Find the velocity function $v(t)$ and the displacement over $[0, 3]$.

**MISTAKE ANALYSIS**

*$v(t) = \int (6t-2)\,dt = 3t^2 - 2t + C$. $v(0) = 3 \Rightarrow C = 3$. $v(t) = 3t^2 - 2t + 3$. Displacement $= \int_0^3 (3t^2 - 2t + 3)\,dt = \left[t^3 - t^2 + 3t\right]_0^3 = 27 - 9 + 9 = 27$ m. Students who use $C = 0$ (forgetting to apply the initial condition) get $v(t) = 3t^2 - 2t$ and a wrong displacement. The constant of integration must always be determined using given conditions. Here $v(0) = 3$ gives $C = 3$.*

**Question 2** Medium [5 marks]

Find the average value of $\sin x$ over $[0, \pi]$.

**MISTAKE ANALYSIS**

$\bar{f} = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\, dx = \frac{1}{\pi}\big[-\cos x\big]_0^{\pi} = \frac{1}{\pi}(-\cos\pi + \cos 0) = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$. *Students who evaluate* $\frac{\sin 0 + \sin\pi}{2} = 0$ *use the trapezoidal rule at the endpoints, which gives zero (since* $\sin 0 = \sin\pi = 0$*) – clearly wrong for a non-negative function. The average value of a positive function on* $[0, \pi]$ *cannot be zero. The formula* $\frac{1}{b-a}\int_a^b f\, dx$ *is the only correct approach.*

**Question 3** Medium [5 marks]

Use the shell method to find the volume of the solid formed when the region bounded by $y = x^2$, $x = 0$, and $x = 2$ is rotated about the $y$-axis.

**MISTAKE ANALYSIS**

*Shell method:* $V = 2\pi\int_0^2 x \cdot x^2\, dx = 2\pi\int_0^2 x^3\, dx = 2\pi\left[\frac{x^4}{4}\right]_0^2 = 2\pi \cdot 4 = 8\pi$. *The shell formula is* $V = 2\pi\int_a^b x\, f(x)\, dx$*: the factor* $x$ *represents the radius of each shell. Students who use the disk method directly (integrating with respect to* $y$ *from* $y = 0$ *to* $y = 4$*) also obtain* $8\pi$ *– the shell and disk methods give the same result when applied correctly. Students who write* $V = \pi\int_0^2 x^2\, dx$ *(using the disk formula with integration along* $x$*) are applying the* $x$*-axis disk formula to a* $y$*-axis rotation – this is wrong.*

**Question 4** Hard [6 marks]

A particle moves with velocity $v(t) = \sin(\pi t)\,\mathrm{m\,s^{-1}}$ for $0 \le t \le 2$. Find the displacement and total distance travelled.

**MISTAKE ANALYSIS**

*Displacement* $= \int_0^2 \sin(\pi t)\, dt = \left[-\frac{\cos(\pi t)}{\pi}\right]_0^2 = -\frac{1}{\pi}(\cos 2\pi - \cos 0) = -\frac{1}{\pi}(1-1) = 0$. $v(t) = 0$*:* $\sin(\pi t) =$

$0 \Rightarrow t = 0, 1, 2$. *On* $[0, 1]$*:* $v > 0$*; on* $[1, 2]$*:* $v < 0$. $\displaystyle\int_0^1 \sin(\pi t)\,dt = \left[-\frac{\cos(\pi t)}{\pi}\right]_0^1 = \frac{1}{\pi}(-\cos\pi + \cos 0) = \frac{2}{\pi}$. *Total distance* $= \frac{2}{\pi} + \frac{2}{\pi} = \frac{4}{\pi}$. *Zero displacement again (oscillation). Distance is* $\frac{4}{\pi} \approx 1.27\,m$. *The pattern: displacement can be zero even when the particle has been moving – it simply returned to its start.*

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### Question 5 Hard [5 marks]

Find the area under $y = \dfrac{1}{x}$ from $x = 1$ to $x = e^2$.

**MISTAKE ANALYSIS**

$A = \displaystyle\int_1^{e^2} \frac{1}{x}\,dx = \left[\ln x\right]_1^{e^2} = \ln(e^2) - \ln 1 = 2 - 0 = 2$. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$*: this is the fundamental integral of* $\frac{1}{x}$*. Students who write* $\frac{1}{x} = x^{-1}$ *and then apply the power rule:* $\int x^{-1}\,dx = \frac{x^0}{0}$ *– division by zero. The power rule* $\int x^n\,dx = \frac{x^{n+1}}{n+1}$ *is undefined for* $n = -1$. $\int \frac{1}{x}\,dx = \ln|x|$ *is the one exception to the power rule.*

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### Question 6 Hard [9 marks]

A particle starts at the origin with velocity $v(t) = 4t - t^2\,\mathrm{m\,s^{-1}}$ for $t \geq 0$.

(a) When does the particle change direction?

(b) Find the position of the particle at $t = 4$ and at $t = 6$.

(c) Find the total distance travelled from $t = 0$ to $t = 6$.

**MISTAKE ANALYSIS**

*(a)* $v = t(4 - t) = 0 \Rightarrow t = 0$ *or* $t = 4$. *The particle changes direction at* $t = 4$ *(from* $v > 0$ *to* $v < 0$*). (b)* $x(t) = \displaystyle\int (4t - t^2)\,dt = 2t^2 - \frac{t^3}{3} + C$. $x(0) = 0 \Rightarrow C = 0$. $x(4) = 32 - \frac{64}{3} = \frac{32}{3}\,m$. $x(6) = 72 - 72 = 0\,m$ *(particle returns to origin). (c) On* $[0, 4]$*:* $v > 0$*, distance* $= x(4) - x(0) = \frac{32}{3}$*. On* $[4, 6]$*:* $v < 0$*, distance* $= |x(6) - x(4)| = |0 - \frac{32}{3}| = \frac{32}{3}$*. Total distance* $= \frac{32}{3} + \frac{32}{3} = \frac{64}{3}\,m$*. Students who compute* $\int_0^6 (4t - t^2)\,dt = 72 - 72 = 0$ *find displacement, not distance. The particle returns to the origin, so displacement is zero, but it has travelled* $\frac{64}{3} \approx 21.3\,m$.

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**M1** method · **A1** accuracy · **R1** reasoning

### Solution – Question 1

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| $v(t) = 3t^2 - 2t + C$; $v(0) = 3 \Rightarrow C = 3$; displacement $= [t^3 - t^2 + 3t]_0^3$ | $v(t) = 3t^2 - 2t + 3$; $27\,\mathrm{m}$ | **M1** |

### Solution – Question 2

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| $\frac{1}{\pi}[-\cos x]_0^\pi = \frac{1}{\pi}(1+1)$ | $\frac{2}{\pi}$ | **M1** |

### Solution – Question 3

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| Shell: $2\pi \int_0^2 x^3\,dx = 2\pi[\frac{x^4}{4}]_0^2 = 2\pi \cdot 4$ | $8\pi$ | **M1** |

### Solution – Question 4

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|---|---|---|
| Displacement $= [-\frac{\cos \pi t}{\pi}]_0^2 = 0$; $v = 0$ at $t = 1$; $\int_0^1 = 2/\pi$ | Displacement $= 0$; distance $= 4/\pi$ | **M1** |

### Solution – Question 5

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| $[\ln x]_1^{e^2} = 2 - 0$ | 2 | **M1** |

### Solution – Question 6

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|---|---|
| (a) $t = 4$ (changes direction) | **M1** |
| (b) $x(t) = 2t^2 - t^3/3$; $x(4) = 32/3$; $x(6) = 0$ | **A1** |
| (c) $32/3 + 32/3$ $64/3\,\text{m}$ | **A1** |

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