THE SANCTUARY OF LEARNING – ABDULWADOOD.ORG

Applications of Integration

Mistake Analysis – Set I

Course	IB Mathematics: Analysis & Approaches HL
Topic	Topic 5 – Calculus: Applications of Integration
Level	Easy → Medium
Questions	6
Marks	35 total. **M1** method · **A1** accuracy · **R1** reasoning.

BEFORE YOU BEGIN

Area between curves: $A = \int_a^b [f(x) - g(x)]\,dx$ where $f(x) \geq g(x)$ on $[a, b]$. Always find intersection points first. If curves cross, split the integral at each crossing.

Volume of revolution about x-axis (disk method): $V = \pi \int_a^b [f(x)]^2\,dx$.

Kinematics: displacement $= \int_a^b v(t)\,dt$; distance $= \int_a^b |v(t)|\,dt$.

Question 1 — Easy — [5 marks]

Find the area enclosed between $y = x^2$ and $y = x$.

MISTAKE ANALYSIS

Intersections: $x^2 = x \Rightarrow x = 0$ or $x = 1$. On $[0, 1]$: $x \geq x^2$, so the upper curve is $y = x$. $A = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$. Students who integrate $x^2 - x$ (wrong order) get $-\frac{1}{6}$ – a negative area. Always sketch the curves and identify the upper function before integrating. If unsure which is on top, test a point: at $x = \frac{1}{2}$: $y = \frac{1}{2} > y = \frac{1}{4}$, so $y = x$ is above $y = x^2$.

Question 2 — Medium — [5 marks]

Find the area enclosed between $y = \sin x$ and $y = \cos x$ on $[0, \frac{\pi}{2}]$.

MISTAKE ANALYSIS

The curves intersect at $x = \frac{\pi}{4}$ *(since* $\sin(\pi/4) = \cos(\pi/4)$*). On* $[0, \pi/4]$*:* $\cos x \geq \sin x$*. On* $[\pi/4, \pi/2]$*:* $\sin x \geq \cos x$*.* $A = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx$*.* $= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{\pi/2}$*.* $= (\sqrt{2} - 1) + (-1 + \sqrt{2}) = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$*. Students who integrate* $\cos x - \sin x$ *over the full interval* $[0, \pi/2]$ *get zero (since the positive and negative parts cancel). The curves cross at* $\pi/4$*, so the integral must be split there.*

Question 3 Medium [5 marks]

Find the volume of the solid formed when the region bounded by $y = x^2$, $x = 0$, $x = 2$, and $y = 0$ is rotated 360 about the x-axis.

MISTAKE ANALYSIS

Disk method: $V = \pi \int_0^2 (x^2)^2\, dx = \pi \int_0^2 x^4\, dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$*. Students who forget to square the function write* $V = \pi \int_0^2 x^2\, dx = \frac{4\pi}{3}$*, which is the volume formula applied to the radius* $f(x)$*, not* $[f(x)]^2$*. The disk formula integrates* $[f(x)]^2$*, because the cross-sectional area of a disk of radius* r *is* πr^2*.*

Question 4 Medium [6 marks]

Find the total area enclosed between $y = x^3$ and $y = x$.

MISTAKE ANALYSIS

Intersections: $x^3 = x \Rightarrow x(x^2 - 1) = 0 \Rightarrow x = -1, 0, 1$*. By symmetry about the origin: total area* $= 2 \times \int_0^1 (x - x^3)\, dx$*.* $= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = 2 \cdot \frac{1}{4} = \frac{1}{2}$*. Students who integrate* $x^3 - x$ *from* -1 *to* 1 *without splitting get* 0 *– the signed areas cancel by symmetry. "Area" is always non-negative: when the curves cross, split and take absolute values.*

Question 5 Medium [6 marks]

Find the volume generated when the region bounded by $y = \sqrt{x}$, $x = 0$, $x = 4$, and $y = 0$ is rotated about the x-axis.

MISTAKE ANALYSIS

$V = \pi \int_0^4 (\sqrt{x})^2\,dx = \pi \int_0^4 x\,dx = \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \cdot 8 = 8\pi$. Note: $(\sqrt{x})^2 = x$, which simplifies the integrand significantly. Students who leave the integrand as $\sqrt{x}$ (forgetting to square) integrate $\pi \int_0^4 x^{1/2}\,dx = \pi \cdot \frac{2}{3} \cdot 8 = \frac{16\pi}{3}$ – the wrong answer. The disk formula squares the radius: $\pi[f(x)]^2$, not $\pi f(x)$.

Question 6 Medium [8 marks]

A particle moves along a straight line with velocity $v(t) = t^2 - 4t + 3\,\mathrm{m\,s^{-1}}$ for $0 \le t \le 3$.

(a) Find the displacement of the particle over the interval $[0, 3]$.

(b) Find the total distance travelled over the interval $[0, 3]$.

MISTAKE ANALYSIS

(a) Displacement $= \int_0^3 (t^2 - 4t + 3)\,dt = \left[\frac{t^3}{3} - 2t^2 + 3t\right]_0^3 = 9 - 18 + 9 = 0\,m$. (b) $v = 0$: $(t-1)(t-3) = 0 \Rightarrow t = 1, 3$. On $[0,1]$: $v > 0$; on $[1,3]$: $v < 0$. $\int_0^1 (t^2 - 4t + 3)\,dt = \frac{1}{3} - 2 + 3 = \frac{4}{3}$. $\left|\int_1^3 (t^2 - 4t + 3)\,dt\right| = \left|0 - \frac{4}{3}\right| = \frac{4}{3}$. Total distance $= \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$ m. The displacement is zero because the particle returns to its starting position. Total distance is not zero – it counts movement in both directions. Students who integrate from 0 to 3 without splitting find displacement (0), not distance (8/3).

M1 method · **A1** accuracy · **R1** reasoning

Solution – Question 1

Intersections $x = 0, 1$; $A = \int_0^1 (x - x^2)\,dx$ | $[\frac{x^2}{2} - \frac{x^3}{3}]_0^1 = \frac{1}{6}$ | **M1**

Solution – Question 2

Split at $x = \pi/4$; $[\sin x + \cos x]_0^{\pi/4} = \sqrt{2} - 1$; $[-\cos x - \sin x]_{\pi/4}^{\pi/2} = \sqrt{2} - 1$ | $2(\sqrt{2} - 1)$ | **M1**

Solution – Question 3

$V = \pi \int_0^2 x^4\,dx$ | $\pi[\frac{x^5}{5}]_0^2 = \frac{32\pi}{5}$ | **M1**

Solution – Question 4

Intersections $x = -1, 0, 1$; symmetry: $2\int_0^1 (x - x^3)\,dx$ | $2[\frac{x^2}{2} - \frac{x^4}{4}]_0^1 = \frac{1}{2}$ | **M1**

Solution – Question 5

$V = \pi \int_0^4 x\,dx$ | $\pi[\frac{x^2}{2}]_0^4 = 8\pi$ | **M1**

Solution – Question 6

(a) $\int_0^3 (t^2 - 4t + 3)\,dt = 0$; (b) $v = 0$ at $t = 1, 3$; split $\int_0^1 v\,dt = 4/3$; $\lvert \int_1^3 v\,dt \rvert = 4/3$; total Displacement $= 0$ (returns to start); distance $\neq 0$	$\dfrac{8}{3}$ m	**M1** **A1** **R1**
